StudentID: ____________________

PHYS 222 Classical Mechanics II (Spring 2019)
Instructor: Sergiy Bubin
Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others).
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that could be be useful may be provided in the appendix. Please look through it *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Among all curves of length l in the upper half-plane passing through the points $(-a, 0)$ and $(a, 0)$ find the one which together with the interval $[-a, a]$ on the x-axis encloses the largest area.

Problem 2. A particle moves in a spherically symmetric potential given by $V(r) = -\frac{k}{r}$.

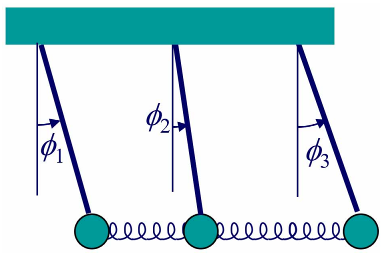
(a) Calculate the Hamiltonian function in spherical coordinates.

(b) Find Hamilton's equations of motion.

(c) Are there any cyclic coordinates in this system? What does it imply?

Problem 3. Consider a methane molecule (CH_4). It has a tetrahedral shape. Assuming that the mass of the hydrogen atom is m, the mass of the carbon atom is $12m$, and the C–H bond length is a, determine the principal moments of inertia in the center-of-mass frame. *Hint: It might be helpful to think about symmetry. The shape of the methane molecule has a very high degree of symmetry. Whatever you do, however, make sure you provide clear explanations.*

Problem 4. Consider three identical pendula coupled by two identical springs as shown in the sketch. Find the normal frequencies and normal modes. Depict graphically the oscillations that correspond to the normal modes of this system.



Problem 5. Two particles are moving along orthogonal axes (say x and y). Each particle moves at a speed v with respect to the same inertial frame. What is the relative velocity of the particles? Assume that v is not small compared to the speed of light c.

Appendix: formula sheet

Lagrangian formalism

$$L = T - V, \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Least action principle

$$\delta S = 0, \qquad S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt.$$

Beltrami identity

$$J[y] = \int_a^b F(y, y')dx, \qquad F - y'\frac{\partial F}{\partial y'} = \text{const}.$$

Hamiltonian formalism

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i p_i \dot{q}_i - L, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \qquad \dot{f} = \frac{\partial f}{\partial t} + \{f, H\}.$$

Canonical transformation

$P = P(p, q),\ Q = Q(p, q)$ is canonical if leaves the form of the Hamilton equations unchanged. Also $\{Q, P\} = 1$.

Liouville's theorem

$$\sum_i \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) + \frac{\partial \rho}{\partial t} = 0.$$

Virial theorem

$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$. For a system with the interaction potential $V(r) = \alpha r^n$ we have $\overline{T} = \frac{n}{2}\overline{V}$

Definition of the tensor of inertia

$$I = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix}$$

Displaced axis theorem

Tensor of inertia about an origin displaced by a constant vector $\mathbf{a}$ is given by
$(I_{\mathbf{a}})_{\alpha\beta} = (I_{\text{c.m.}})_{\alpha\beta} + M(a^2\delta_{\alpha\beta} - a_\alpha a_\beta)$

Moment of inertia about an axis defined by a normal vector

Moment of inertia $I_{\mathbf{n}}$ about an axis that passes through the center of mass and is defined by a normal vector $\mathbf{n}$ is given by (here I is the tensor of inertia of the system):
$I_{\mathbf{n}} = \mathbf{n}^T I \mathbf{n}$

Euler angles and the transformation between the lab and body frame

If $\mathbf{r}'$ is the position in the fixed/lab frame and $\mathbf{r}$ is the position in the body frame then $\mathbf{r} = U_\psi U_\theta U_\phi \mathbf{r}' = U\mathbf{r}'$, where

$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad U_\phi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler equations for a rigid body

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

Normal frequencies and normal modes of a system of n coupled harmonic oscillators

Generalized eigenvalue problem: $K\mathbf{a}^{(i)} = \omega_i^2 M\mathbf{a}^{(i)}$
Trajectories: $\mathbf{x}(t) = \sum\limits_i \mathbf{a}^{(i)} \mathrm{Re}[e^{i\omega_i t}]$

Unifom continuous string of length L

The equation of motion (wave equation): $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad 0 \le x \le L \quad v = \sqrt{\tau/\rho}$
Normal frequencies: $\omega_n = \frac{n\pi v}{L}$
General solution: $\sum\limits_{n=1}^{\infty} (\beta_n \cos\omega_n t + \gamma_n \sin\omega_n t)\sin\frac{n\pi x}{L}$

Orthogonality of $\sin$ and $\cos$ functions on $(0, L)$ interval

$$\int_0^L \sin\tfrac{n\pi x}{L}\,\sin\tfrac{m\pi x}{L}\,dx = \tfrac{L}{2}\delta_{nm} \qquad \int_0^L \cos\tfrac{n\pi x}{L}\,\cos\tfrac{m\pi x}{L}\,dx = \tfrac{L}{2}\delta_{nm} \qquad \int_0^L \sin\tfrac{n\pi x}{L}\,\cos\tfrac{m\pi x}{L}\,dx = 0$$

Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

Lorentz transform

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Transformation of velocities in Special Relativity

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \qquad u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)} \qquad u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)} \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Useful integrals

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right) + C$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arctan\left[\frac{x}{\sqrt{a^2 - x^2}}\right]\right) + C$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \ne 0$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C, \quad a \ne 0$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C = \operatorname{arccosh} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{a} + C, \quad a \neq 0$$