An alternative form of the Euler-Lagrange equation
(Beltrami equation)

Previously we showed that a necessary condition for functional

$$J[y(x)] = \int_{x_1}^{x_2} F(y, y', x)\,dx$$

to have an extremum value is the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

If the integrand F in J does not depend explicitly on x, the Euler-Lagrange equation can be recast in an alternative and simpler form known as the Beltrami equation. Let us present it here.

First let us note that the total derivative of $F - \frac{dF}{dy'}y'$ with respect to x can be written as

$$\frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) = \frac{\partial F}{\partial y}\frac{dy}{dx} + \cancel{\frac{\partial F}{\partial y'}\frac{dy'}{dx}} - \left(\frac{d}{dx}\frac{\partial F}{\partial y'}\right)y' - \cancel{\frac{\partial F}{\partial y'}\frac{dy'}{dx}}$$

The second and fourth terms cancel. Now if we make use the Euler-Lagrange equation $\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}$ we can see immediately that the other two terms cancel too.

Thus we end up with

$$\frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) = 0$$

which means

$$F - \frac{\partial F}{\partial y'}y' = \text{const} \quad \leftarrow \text{Beltrami equation}$$

In contrast to the original Euler-Lagrange equation, the Beltrami equation is first-order. It is typically much easier to deal with

To illustrate how it works let us again consider a problem of finding a curve $y(x)$ of minimal length that connects points (x_1, y_1) and (x_2, y_2). The functional to minimize is

$$I[y(x)] = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

The Beltrami equation gives us

$$\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} = \text{const}$$

which is satisfied if

$$y' = \text{const} \quad \text{or} \quad y = C_1 x + C_2 = \frac{y_2 - y_1}{x_2 - x_1} x + \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$

Constrained extremization problem

In certain problems, while extremizing a given functional $J[y]$, along with the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ we may also need the extremizing function to satisfy an additional integral constraint:

$$J[y] = \int_{x_1}^{x_2} F(y, y', x)\, dx$$

while

$$\int_{x_1}^{x_2} G(y, y', x)\, dx = \ell = \text{const} \quad \leftarrow \text{constraint}$$

Just like in the case of finding constrained extrema of functions, we can convert the constrained problem to an unconstrained one by the Lagrange multiplier technique.
It amounts to defining a new functional

$$I[y] = J[y] + \lambda \int_{x_1}^{x_2} G(y, y', x)\, dx$$

with the end point conditions $y(x_1) = y_1 \quad y_2(x_2) = y_1$
and optimizing $I[y]$ without constraints, i.e. finding

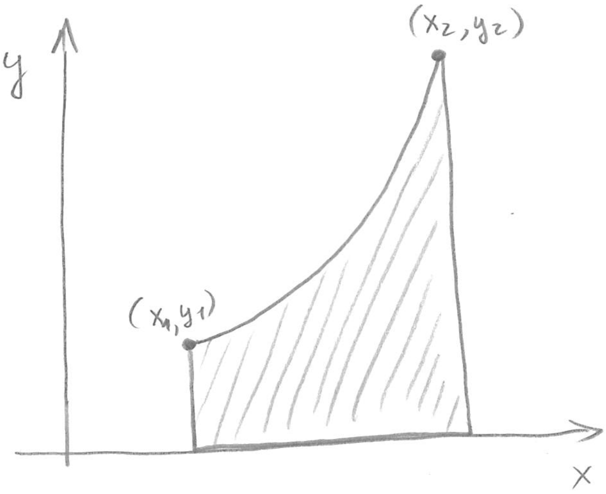
such $y(x)$ and λ that extremize $I[y]$.

As an example, let us consider the Dido's problem:
Find the curve of fixed perimeter which has maximum area above the x-axis

$$J[y(x)] = \int_{x_1}^{x_2} y(x)\,dx \qquad \text{(area)}$$

$$\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx = \ell \qquad \text{(perimeter)}$$

$y(x_1) = y_1$ and $y(x_2) = y_2$



In this case

$$F(y, y', x) = y(x)$$
$$G(y, y', x) = \sqrt{1+y'^2}$$

Let us define $H(y,y',x) = F(y,y',x) + \lambda G(y,y',x) =$
$$= y + \lambda\sqrt{1+y'^2}$$

The problem of maximizing the functional
$$\int_{x_1}^{x_2} H(y,y',x)\,dx$$
is solved by solving the Euler-Lagrange equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\frac{\partial H}{\partial y'} = 0 \qquad y(x_1) = y_1 \quad y(x_2) = y_2$$

The derivatives are

$$\frac{\partial H}{\partial y} = 1 \qquad \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1+y'^2}}$$

so our Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 1 \implies \frac{\lambda y'}{\sqrt{1+y'^2}} = x - a$$

or $\lambda^2 y'^2 = (1+y'^2)(x-a)^2$

and

$$y'^2\left(\lambda^2-(x-a)^2\right) = (x-a)^2$$

$$y' = \frac{x-a}{\sqrt{\lambda^2-(x-a)^2}}$$

Integrating the last expression yields

$$y(x) = -\sqrt{\lambda^2-(x-a)^2} + b \qquad (a, b = \text{const})$$

We can also express it as

$$(x-a)^2 + (y-b)^2 = \lambda^2$$

which is an equation that describes a circle. Constants a, b, and λ can be obtained from

$$y(x_1) = y_1 \qquad y(x_2) = y_2 \qquad \text{and} \qquad \ell = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$$

which provide a system of three algebraic equations

Extending the least action principle to systems with constraints

Using the method of Lagrange multipliers we can account for constraints in the action integral

$$S = \int_{t_1}^{t_2} L\, dt \quad \longrightarrow \quad I = \int_{t_1}^{t_2} \left(L + \sum_{j=1}^{m} \lambda_j g_j\right) dt$$

In this case we do not need to worry about the independence of all q_i's (generalized coordinates). Instead we are allowed to vary all q_i's and all λ_j's independently. The variations of λ_j's give the m constraint equations:

$$g_j(q_1, \dots, q_n, t) = 0 \qquad j = 1, \dots, m$$

Here we only consider holonomic constraints (those that do not depend on generalized velocities $\dot{q}_i$.)

The variations of q_i's give

$$\delta I = \int_{t_1}^{t_2} dt \left(\sum_{i=1}^{n} \left\{ \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m}{}' \lambda_j \frac{\partial q_j}{\partial q_i} \right\} \delta q_i \right) = 0$$

$$\left(\text{recall : } \delta J = \int \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta y \, dx \quad \text{for} \quad J = \int F(y, y', x)\, dx \right)$$

with no derivatives $\frac{\partial q_j}{\partial \dot{q}_i}$ are present in the sum over j because the constraints are holonomic. Now, since not all δq_i are independent, we choose λ_j's in such a way that m of the equations are satisfied for arbitrary δq_i and then choose the variations δq_i in the remaining $n-m$ equations independently. Then we obtain

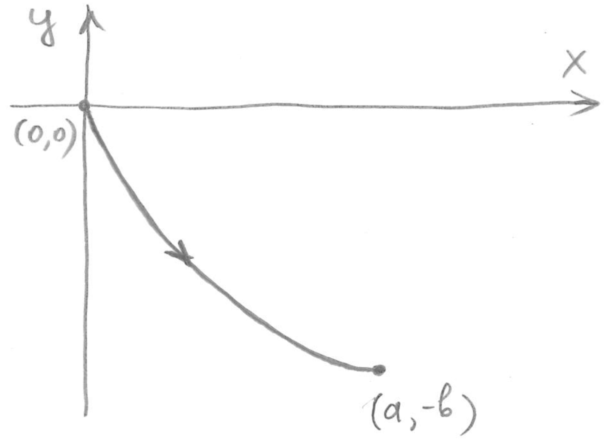
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m}{}' \lambda_j \frac{\partial q_j}{\partial q_i} = 0 \qquad i = 1..m$$

The equality follows from the choice of λ_js. We also have the same equations for $j = m+1, .., n$ where the equality follows from the virtual variations of the δq_i

The Brachistochrone problem

A classical problem in the calculus of variations is the brachistochrone problem that reads as follows:
Find a path that allows a particle (subject to gravity force) to travel between two points (x_1, y_1) and (x_2, y_2) in a vertical plane in the shortest possible time

For convenience we can choose point (x_1, y_1) to be the origin of our coordinate system $(0,0)$. Let us denote the second end point $(a,-b)$ $a,b>0$



While the straight line connecting the two end points would have the shortest lenght it is not the path that corresponds to the shortest travel time. This is because the particle (of mass m) is accelerated by the gravity force. The lower its vertical position the faster it moves — the kinetic energy is converted from the potential energy.
To solve the problem we first find the time the particle needs to arrive at the second end point as a functional of the curve $y(x)$. The length of each segment of this curve is given by

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'(x)^2}\, dx$$

The time dt the particle spends traveling this segment is $dt = \frac{dl}{v}$ where v is the speed.

The speed can be inferred from the conservation of energy:

$$\frac{m v^2}{2} + mgy = 0 \qquad (\text{we start at } y=0)$$

$$v = \sqrt{-2gy} \qquad (y<0)$$

Then

$$dt = \sqrt{\frac{1+y'(x)^2}{-2g\,y(x)}}\, dx$$

and the total time is given by the integral

$$t[y(x)] = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1+y'^2}{-y}}\, dx$$

Now let us use the alternative form of the Euler-Lagrange equation (Beltrami equation) to minimize our time functional $t[y]$:

$$F(y,y') = \sqrt{\frac{1+y'^2}{-y}}$$

$$F - \frac{\partial F}{\partial y'} y' = C \qquad (C = \text{const})$$

$$\sqrt{\frac{1+y'^2}{-y}} - \frac{y'}{\sqrt{1+y'^2}\,\sqrt{-y}}\, y' = C$$

$$\frac{1}{\sqrt{-y(1+y'^2)}} = C$$

or

$$1 + y'^2 = -\frac{2R}{y} \qquad \text{where} \quad R = \frac{1}{2C^2}$$

and

$$y' = \pm \sqrt{\frac{2R+y}{-y}}$$

In the last equation we choose the minus sign as we expect the particle to move down (at least initially)

Then we can integrate

$$-\int_0^{y(x)} \sqrt{\frac{-y}{2R+y}}\, dy = \int_0^x d\tilde{x}$$

The integral can be taken with a substitution

$$y = -2R \sin^2\frac{\theta}{2} \qquad dy = -2R \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta$$

$$\sqrt{2R+y} = \sqrt{2R\left(1-\sin^2\frac{\theta}{2}\right)} = \sqrt{2R}\cos\frac{\theta}{2}$$

so that we get

$$2R \int_0^{\theta(x)} \sin^2\frac{\theta}{2}\, d\theta = R \int_0^{\theta(x)} (1-\cos\theta)\, d\theta = x$$

$$R\left(\theta(x) - \sin\theta(x)\right) = x$$

It is easier to represent the solution curve in parametrized form, having both x and y a functions of θ:

$$\begin{cases} x(\theta) = R(\theta - \sin\theta) \\ y(\theta) = -2R\sin^2\frac{\theta}{2} = -R(1-\cos\theta) \end{cases}$$

These parametric equations define a curve called cycloid. Constant R can be adjusted in such a way that the cycloid passes through the second end point at $(a,-b)$.