The virial theorem.

If the motion takes place in a finite region of space we can establish a simple and useful relation between the time average values of the kinetic and potential energies, known as the virial theorem. This theorem is statistical in nature.

Let us consider a general system of particles with coordinates $\vec{r}_i$ and applied forces $\vec{F}_i$ (including the forces of constraint). For each particle

$$\dot{\vec{p}}_i = \vec{F}_i$$

If we introduce the quantity $G = \sum_i \vec{p}_i \cdot \vec{r}_i$, its total time derivative is

$$\frac{dG}{dt} = \underbrace{\sum_i \dot{\vec{r}}_i \cdot \vec{p}_i}_{= \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = \sum_i m_i v_i^2 = 2T} + \underbrace{\sum_i \dot{\vec{p}}_i \cdot \vec{r}_i}_{= \sum_i \vec{F}_i \cdot \vec{r}_i}$$

Then

$$\frac{d}{dt} \sum_i \vec{p}_i \cdot \vec{r}_i = 2T + \sum_i \vec{F}_i \cdot \vec{r}_i$$

Time averaging the latter equation over a sufficiently long time interval τ yields

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt \equiv \underbrace{\overline{\frac{dG}{dt}}}_{\frac{1}{\tau}\left[G(\tau)-G(0)\right]} = \overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

If the motion is periodic and τ is chosen to be the period then $G(\tau) - G(0) = 0$. But even if the motion is not exactly periodic (just finite) and $\tau \to \infty$ the left-hand side vanishes. Then

$$\overline{T} = \underbrace{-\frac{1}{2}\,\overline{\sum_i \vec{F}_i \cdot \vec{r}_i}}_{\text{virial of Clausius}} \quad \leftarrow \text{virial theorem}$$

When the forces are derivable from a potential the theorem becomes

$$\overline{T} = \frac{1}{2}\,\overline{\sum_i \frac{\partial V}{\partial \vec{r}} \cdot \vec{r}}$$

In the case of a single particle moving in a central field

$$\overline{T} = \frac{1}{2}\,\overline{\frac{\partial V}{\partial r}\, r}$$

The virial theorem is particularly useful when V is a power-law function, $V = \alpha r^k$. In this case

$$\frac{\partial V}{\partial r}\, r = k V$$

and

$$\overline{T} = \frac{k}{2}\,\overline{V}$$

In particular, for a harmonic oscillator potential $(k = 2)$ we get $\overline{T} = \overline{V}$, while for the Coulomb potential $\overline{T} = -\frac{1}{2}\,\overline{V}$

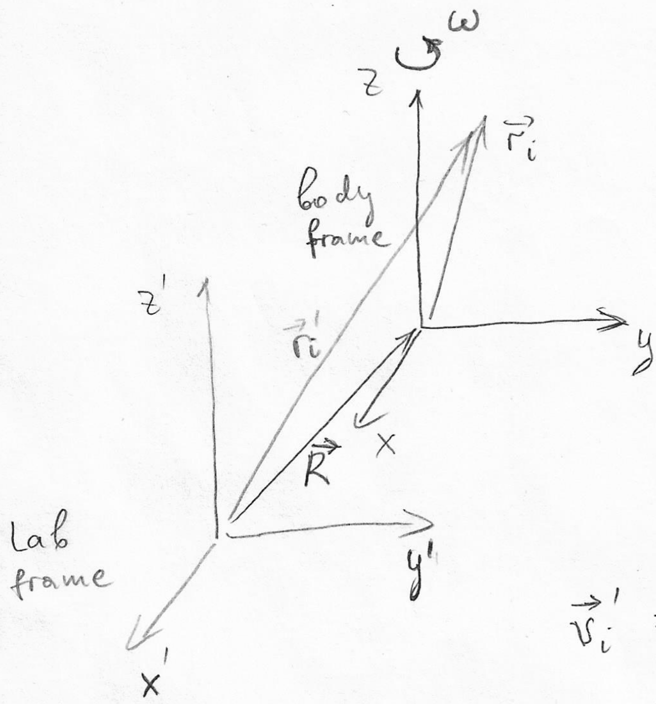
It should be noted that the applications of the virial theorem are not limited to classical mechanics. It finds its way to quantum mechanics as well

Review of basic results for rotational motion of rigid bodies

A rigid body is a collection of N particles with the property that the distances between any of its constituent particles are fixed. While the arbitrary system requires $3N$ coordinates to specify its configuration, the rigid body requires only six such coordinates.

Since all interportiele distances in a rigid body are fixed, the internal potential energy, $V^{int} = \sum_{ij}' V_{ij}(\vec{r}_{ij})$, is a constant and can be dropped from consideration.

Let us consider a rotating rigid body. K' and K are coordinate systems that correspond to the laboratory frame of reference and the frame of reference attached to our rigid body. Here we assume that K may be rotating but there is no translational motion within K.



In the lab frame the velocity of the ith particle is

$$\vec{v}_i' = \dot{\vec{R}} + \vec{\omega} \times \vec{r}_i$$

For an observer in the lab frame the total kinetic energy is

$$T' = \frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2}\underbrace{\left(\sum_i m_i\right)}_{M}\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\left(\vec{\omega}\times\underbrace{\sum_i m_i\vec{r}_i}_{M\cdot\vec{r}_{cm}}\right)$$

$$+ \frac{1}{2}\sum_i m_i\left(\vec{\omega}\times\vec{r}_i\right)^2$$

A great simplification results if we choose K to be the center of mass, so that $\vec{r}_{CM} = 0$

With this choice

$$T_{trans} = \frac{1}{2} M \dot{\vec{R}}^2$$

$$T' = \underbrace{T_{transl.}}_{\text{motion of the system as a whole}} + \underbrace{T_{rot}}_{\text{rotation about the C.M.}} \qquad \text{where} \qquad T_{rot} = \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2$$

From now on we will assume that K is located at the center of mass. Let us turn our attention to the rotation in the K frame. Using the Binet-Cauchy vector identity

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D})$$

we can write

$$(\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i) = \vec{\omega} \cdot \left(r_i^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) \vec{r}_i \right) = r_i^2 \omega^2 - (\vec{r}_i \cdot \vec{\omega})(\vec{r}_i \cdot \vec{\omega})$$

The rotational part of the kinetic energy is then

$$T_{rot} = \frac{1}{2} \sum_i m_i \left(\sum_{\alpha=1}^{3} r_{i\alpha}^2 \sum_{\beta=1}^{3} \omega_\beta \omega_\beta - \sum_{\alpha=1}^{3} r_{i\alpha} \omega_\alpha \sum_{\beta=1}^{3} r_{i\beta} \omega_\beta \right)$$

we can also write it as

$$T_{rot} = \frac{1}{2} \sum_{\alpha,\beta} \omega_\alpha \omega_\beta I_{\alpha\beta} = \frac{1}{2} \vec{\omega}^T I \vec{\omega}$$

where

$$I_{\alpha\beta} = \sum_i m_i \left(r_i^2 \delta_{\alpha\beta} - r_{i\alpha} r_{i\beta} \right) \quad \leftarrow \text{tensor of inertia}$$

or, explicitly

$$I = \begin{pmatrix} \sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\ -\sum m_i x_i y_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\ -\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (x_i^2 + y_i^2) \end{pmatrix}$$

The concept of the tensor of inertia can be generalized to the case of continuous mass distribution.

For a point mass dm

$$dI = dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \qquad dm = \rho(\vec{r})\,dx\,dy\,dz$$

Then, for example,

$$I_{12} = -\int_V xy\,\rho(\vec{r})\,dx\,dy\,dz \qquad I_{33} = \int_V (x^2+y^2)\,\rho(\vec{r})\,dx\,dy\,dz$$

Since the tensor of inertia is symmetric, $I_{\alpha\beta} = I_{\beta\alpha}$, there are only six independent components. By choosing a proper orthogonal transformation (i.e. by rotating the K frame by some angle) it is possible to diagonalize I; so that

$$UIU^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

I_1, I_2, and I_3 are called the principal moments. They are all positive (non-negative)

Displaced axis theorem is a generalization of the familiar parallel axis theorem ($I = I_{cm} + Ma^2$). It gives the tensor of inertia about an origin displaced by a constant vector $\vec{a}$:

$$I_{\vec{a}} = I_{cm} + M\left(a^2\delta_{\alpha\beta} - a_\alpha a_\beta\right)$$

Displaced axis theorem is a generalization of the familiar parallel axis theorem ($I = I_{cm} + Ma^2$) and it can be used to compute the tensor of inertia about an origin displaced by a constant vector $\vec{a}$:

$$\vec{r}\,' = \vec{r} + \vec{a}$$

$$I_{\alpha\beta} = \sum_i m_i \left[\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right]$$

$$I'_{\alpha\beta} = \sum_i m_i \left[\delta_{\alpha\beta} (\vec{r}_i + \vec{a})^2 - (r_{i\alpha} + a_\alpha)(r_{i\beta} + a_\beta)\right] =$$

$$= \sum_i m_i \left[\delta_{\alpha\beta} \left(r_i^2 + 2\vec{r}_i \vec{a} + a^2\right) - \left(r_{i\alpha} r_{i\beta} + r_{i\alpha} a_\beta + r_{i\beta} a_\alpha + a_\alpha a_\beta\right)\right] =$$

$$= I_{\alpha\beta} + \sum_i m_i \left[\delta_{\alpha\beta} (2\vec{r}_i \cdot \vec{a} + a^2) - (r_{i\alpha} a_\beta + r_{i\beta} a_\alpha + a_\alpha a_\beta)\right] =$$

$$= I_{\alpha\beta} + \sum_i m_i \left[\delta_{\alpha\beta} a^2 - a_\alpha a_\beta\right]$$

vanish if $\sum_i m_i \vec{r}_i = 0$

$$= I_{\alpha\beta} + M\left[\delta_{\alpha\beta} \vec{a}^2 - a_\alpha a_\beta\right]$$

Now let us consider the angular momentum of a rigid body. Again, we will use body coordinates with the center of mass at the origin.

$$\vec{r}_i{}' = \vec{R} + \vec{r}_i$$

$$\vec{v}_i{}' = \vec{V}_{cm} + \vec{\omega} \times \vec{r}_i \qquad \vec{V}_{CM} \equiv \dot{\vec{R}}$$

$$\vec{L}' \equiv \sum_i \vec{r}_i{}' \times \vec{p}_i{}' = \sum_i m_i (\vec{r}_i{}' \times \vec{v}_i{}') = \sum_i m_i \left((\vec{R} + \vec{r}_i) \times (\vec{V}_{cm} + \vec{\omega} \times \vec{r}_i) \right)$$

$$= \vec{R} \times \vec{P} + \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \qquad \left[\text{recall } \sum_i m_i \vec{r}_i = 0 \right]$$

where $\vec{P} = M\vec{V}_{CM}$. Hence

$$\vec{L}' = \vec{L}_{CM} + \vec{L}_{rot} \qquad \text{with} \quad \vec{L}_{CM} = \vec{R} \times \vec{P}$$

The second term can be simplified using the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\vec{L}_{rot} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_i m_i \left[r_i^2 \vec{\omega} - (\vec{\omega} \cdot \vec{r}_i) \vec{r}_i \right], \text{ or}$$

$$L_{rot_\beta} = \sum_i m_i \left[r_i^2 \omega_\beta - (\vec{\omega} \cdot \vec{r}_i) r_{i\beta} \right] = \sum_i m_i \left[\sum_\gamma r_{i\gamma}^2 \omega_\beta - \sum_\alpha \omega_\alpha r_{i\alpha} r_{i\beta} \right] =$$

$$= \sum_i m_i \left[\sum_\alpha \delta_{\alpha\beta} \sum_\gamma r_{i\gamma}^2 \omega_\alpha - \sum_\alpha \omega_\alpha r_{i\alpha} r_{i\beta} \right] = \sum_\alpha \omega_\alpha \underbrace{\sum_i m_i \left[\delta_{\alpha\beta} \sum_\gamma r_{i\gamma}^2 - r_{i\alpha} r_{i\beta} \right]}_{\to I_{\alpha\beta} = I_{\beta\alpha}}$$

$$L_{rot,\beta} = \sum_\alpha I_{\beta\alpha} \omega_\alpha \qquad \text{or} \qquad \vec{L} = I \vec{\omega}$$

It should be noted that in general $\vec{L}_{rot}$ and $\vec{\omega}$ are not aligned with each other