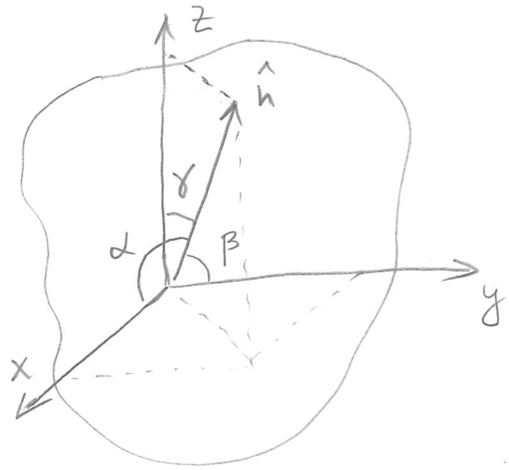
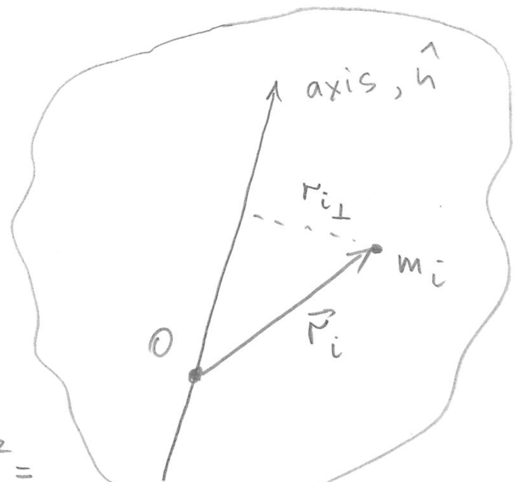
Moment of inertia about an axis defined by a normal vector

Let us relate the tensor of inertia in some coordinate frame with the center of mass at its origin and the moment of inertia about some arbitrary axis that passes through the center of mass.
We will define the direction of the arbitrary axis using a unit vector:



$$\hat{n} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma$$

The moment of inertia is a sum of all particles that constitute our rigid body:



$$I_{\hat{n}} = \sum_i m_i r_{i\perp}^2$$

where $r_{i\perp} = |r_i \sin\varphi| = |\vec{r}_i \times \hat{n}|$

$$r_{i\perp}^2 = |\vec{r}_i \times \hat{n}|^2 = \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix} \right)^2 =$$

$$= (y_i\cos\gamma - z_i\cos\beta)^2 + (z_i\cos\alpha - x_i\cos\gamma)^2 + (x_i\cos\beta - y_i\cos\alpha)^2 =$$

$$= (y_i^2 + z_i^2)\cos^2\alpha + (z_i^2 + x_i^2)\cos^2\beta + (x_i^2 + y_i^2)\cos^2\gamma - 2y_iz_i\cos\beta\cos\gamma - 2z_ix_i\cos\gamma\cos\alpha -$$

$$- 2x_iy_i\cos\alpha\cos\beta$$

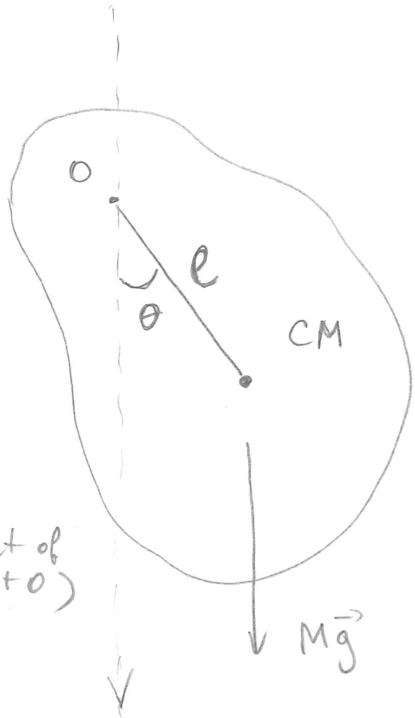
Then

$$m_i r_{i\perp}^2 = (\cos\alpha \;\; \cos\beta \;\; \cos\gamma) \begin{pmatrix} I_{xx}^i & I_{xy}^i & I_{xz}^i \\ I_{yx}^i & I_{yy}^i & I_{yz}^i \\ I_{zx}^i & I_{zy}^i & I_{zz}^i \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} = \hat{n}^T I^i \hat{n}$$

and $I_{\hat{n}} = \sum_i m_i r_{i\perp}^2 = \hat{n}^T I \hat{n}$, where I is the tensor of inertia

Physical pendulum

Let us consider a rigid body of an arbitrary shape that is free to swing under its own weight about a fixed horizontal axis of rotation. It is known as physical pendulum.



The equation that desribes the motion can be obtained either with the Lagrangian method or by considering torques. Let us do that in the former approach. The kinetic energy of rotation is

$$T = \frac{1}{2} I \omega^2 = \frac{1}{2} I \dot{\theta}^2 \quad (I \text{ is the moment of inertia about point } O)$$

The potential energy is given by

$$V = -Mgl\cos\theta$$

It is easy to see that the potential energy is determined by the vertical displacement of the center of mass

$$V = -\sum_i m_i g y_i = -g \sum_i m_i y_i = -g \left(\sum_i m_i\right) \underbrace{\frac{\sum_i m_i y_i}{\sum_i m_i}}_{y_{cm}} = -Mg y_{cm}$$

$$= -Mgl\cos\theta$$

Then

$$L = T - V = \frac{1}{2} I \dot{\theta}^2 + Mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -Mgl\sin\theta$$

and the Lagrange equation is $I\ddot{\theta} + Mgl\sin\theta = 0$

In the limit of small oscillations $\sin\theta \approx \theta$ and we get a harmonic oscillator

$$\ddot{\theta} + \frac{Mg\ell}{I}\theta = 0 \qquad \theta(t) = A\cos(\omega t + \varphi)$$

$$\omega = \sqrt{\frac{Mg\ell}{I}}$$

Now let us see what happens with torques

$$\vec{N}_{tot} = \sum_i \vec{r}_i \times m_i\vec{g} = \left(\sum m_i\vec{r}_i\right)\times\vec{g} = M\vec{r}_{cm}\times\vec{g}$$

When we project $\vec{N}_{tot}$ onto the axis perpendicular to the motion we get

$$N_z = -M\underbrace{r_{cm}}_{\ell}\, g\sin\theta$$

The fundamental equation $\vec{N} = \frac{d\vec{L}}{dt}$ then gives

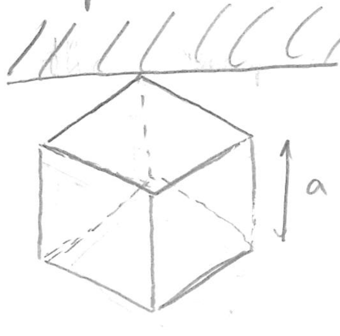
$$-M\ell g\sin\theta = \frac{dL_z}{dt}$$

$$-M\ell g\sin\theta = \frac{d}{dt}I\dot{\theta}$$

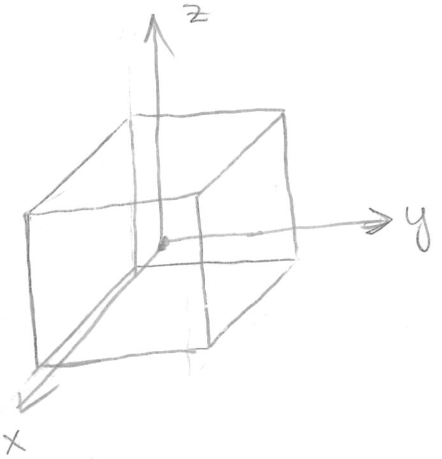
ant it results in the same equations we obtained before:

$$I\ddot{\theta} + M\ell g\sin\theta = 0$$

Let us consider an illustrative example: swinging of a cube of uniform density that is suspended at one of its angles. For that we need to find its moment of inertia (about an axis that goes trough its corner and lies in a plane parallel to the ceiling



But first calculate the moment of inertia a simpler configuration, when the egdes of the cube are aligned along the x, y, and z axes:



$$I_{xx} = \int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \rho (y^2+z^2)\, dx\,dy\,dz$$

$$= \rho a^2 \cdot 2 \cdot \frac{y^3}{3}\Big|_{-a/2}^{a/2} = \frac{\rho a^5}{6} = I_{yy} = I_{zz}$$

$$I_{xy} = \iiint \rho(-xy)\, dx\,dy\,dz = 0$$

In this setting the tensor of inertia is proportional to a unit matrix

$$I = \begin{pmatrix} \frac{\rho a^5}{6} & 0 & 0 \\ 0 & \frac{\rho a^5}{6} & 0 \\ 0 & 0 & \frac{\rho a^5}{6} \end{pmatrix} = \frac{\rho a^5}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The moment of inertia about any specific axis that goes through the center and given by the normal vector $\hat{n}$ is

$$I_{cm} = \hat{n}^T I \hat{n}$$ (* see comment in the end)

Now since I is a unit matrix and $\hat{n}$ is a normalized unit vector

$$I_{cm} = I_{xx} = I_{yy} = I_{zz} = \frac{\rho a^5}{6} = \frac{Ma^2}{6}$$ for any choice of $\hat{n}$

It should be noted that this happens for a cube but not for any coboid where $a_x \neq a_y \neq a_z$

Now we can use the parallel axis theorem to determine the moment of inertia about the point located in the corner:

$$I_{susp} = I_{CM} + Ml^2 = \frac{Ma^2}{6} + M\left(\frac{a\sqrt{3}}{2}\right)^2 =$$

$$= Ma^2\left(\frac{1}{6} + \frac{3}{4}\right) = \frac{11}{12} Ma^2$$

With that we can determine the angular frequency of oscillations

$$\omega = \sqrt{\frac{Mgl}{I}} = \sqrt{\frac{Mg\frac{a\sqrt{3}}{2}}{\frac{11}{12}Ma^2}} = \sqrt{\frac{6\sqrt{3}}{11}\frac{g}{a}}$$