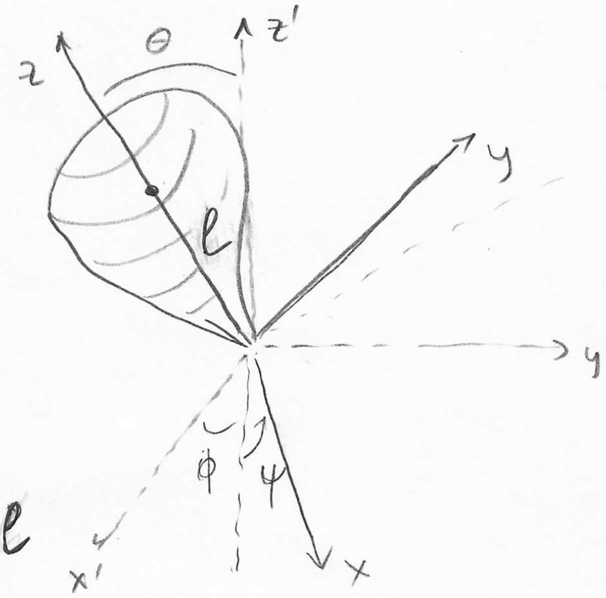
Motion of a symmetric top with one point fixed

Let us consider another example of rigid body rotation — a symmetric top $I_1 = I_2 \neq I_3$ in a uniform gravitational field when one point of the symmetry axis is fixed in space. A wide variety of physical systems, ranging from a child's top to gyroscopes are approximated by such a model.

The configuration of the body is specified by three Euler's angles. The distance of the center of mass (located on the symmetry axis) from the fixed point is denoted by l



θ - inclination of the z axis from the vertical
ϕ - azimuth of the top about the vertical
ψ - rotation of the top about its own z-axis

Let us find the Lagrangian of the system in terms of the Euler's angles. The kinetic energy is

$$T = \frac{1}{2} \sum_i I_i \omega_i^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$$

According to our previous consideration of the Euler's angles, the individual components ω_i in the body frame are

$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$

$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$

$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$

Then

$$\omega_1^2 = \dot{\phi}^2 \sin^2\theta \sin^2\psi + 2\dot{\phi}\dot{\theta} \sin\theta \sin\psi \cos\psi + \dot{\theta}^2 \cos^2\psi$$

$$\omega_2^2 = \dot{\phi}^2 \sin^2\theta \cos^2\psi - 2\dot{\phi}\dot{\theta} \sin\theta \sin\psi \cos\psi + \dot{\theta}^2 \sin^2\psi$$

so that $\omega_1^2 + \omega_2^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2$

and

$$\omega_3^2 = (\dot{\phi}\cos\theta + \dot{\psi})^2$$

With that

$$T = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2$$

The potential energy is $Mgl\cos\theta$ and the Lagrangian becomes

$$L = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$

Both ϕ and ψ are cyclic coordinates. The corresponding conjugated momenta are integrals of motion.

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot{\phi} + I_3 \dot{\psi} \cos\theta = \text{const}$$

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi}\cos\theta) = \text{const}$$

These conjugated momenta are actually the orbital momenta along the $\vec{\phi}$ and $\vec{\psi}$ angles. The gravitational torque is directed along the line of nodes and can have no component along either z or z' axes. The equations for P_ϕ and P_ψ can be solved for $\dot{\phi}$ and $\dot{\psi}$. For $\dot{\psi}$ we can write

$$\dot{\psi} = \frac{p_\psi - I_3 \dot{\phi} \cos\theta}{I_3}$$

Substituting this into the equation for p_ϕ gives

$$(I_1 \sin^2\theta + I_3 \cos^2\theta)\dot{\phi} + (p_\psi - I_3 \dot{\phi} \cos\theta)\cos\theta = p_\phi$$

or

$$I_1 \sin^2\theta \, \dot{\phi} + p_\psi \cos\theta = p_\phi$$

so that

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

Similarly $\dot{\psi}$ can be written as

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos\theta)\cos\theta}{I_1 \sin^2\theta}$$

Our system is conservative. Therefore the total energy should be an integral of motion

$$E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + Mgl\cos\theta = \text{const}$$

Using the expression for ω_3, i.e. $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ we note that the equation for p_ψ can be written as

$$p_\psi = I_3 \omega_3 = \text{const} \qquad \text{or} \qquad I_3 \omega_3^2 = \frac{p_\psi^2}{I_3} = \text{const}$$

Therefore, not only E is an integral of motion but also $E - \frac{1}{2} I_3 \omega_3^2$. We denote it through E'

$$E' \equiv E - \frac{1}{2} I_3 \omega_3^2 = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + Mgl\cos\theta = \text{const}$$

Substituting into this equation the expression for $\dot{\phi}$ above we have

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl\cos\theta$$

The latter can be written as

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

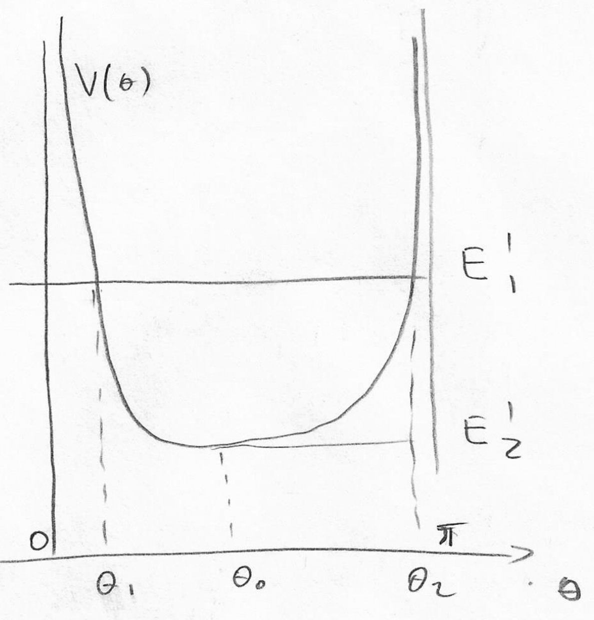
where $V(\theta)$ is an effective potential

$$V(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl \cos\theta$$

The equation for E' can be solved (at least formally) to yield $t(\theta)$

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}[E' - V(\theta)]}}$$

and inverted to obtain $\theta(t)$. Then, this $\theta(t)$ can be substituted into the equations for $\dot{\phi}$ and $\dot{\psi}$ to yield $\phi(t)$ and $\psi(t)$. That would constitute a complete solution of the problem. The procedure itself is not very illuminating, however. But we can deduce some qualitative features of the motion by examining the effective potential in a manner analogous to that used for treating the motion of a particle in a central-force field.



θ_1, θ_2 are turning points when $E' = E'_1$

The inclination of the rotating top is confined within $\theta_1 \leq \theta \leq \theta_2$

When $E' = E'_2 = V_{min}$ θ takes only a single value θ_0

In this case the motion is similar to the occurence of circular orbits in the central-force problem θ_0 can be found from

$$\left.\frac{\partial V}{\partial \theta}\right|_{\theta=\theta_0} = 0$$

$$\frac{-\cos\theta_0 \left(p_\phi - p_\psi \cos\theta_0 \right)^2 + p_\psi \sin^2\theta_0 \left(p_\phi - p_\psi \cos\theta_0 \right)}{I_1 \sin^3\theta_0} - Mgl\sin\theta_0 = 0$$

If we define $\beta = p_\phi - p_\psi \cos\theta_0$ then it becomes

$$\cos\theta_0\, \beta^2 - p_\psi \sin^2\theta_0\, \beta + Mgl I_1 \sin^4\theta_0 = 0$$

This quadratic equation can be (formally) solved for β:

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MglI_1\cos\theta_0}{p_\psi^2}} \right)$$

Now β must be a real quantity. Therefore the expression in the radical must be positive

If $\theta_0 < \frac{\pi}{2}$ then

$$p_\psi^2 \geq 4MglI_1\cos\theta_0$$

However we had it previously that $p_\psi = I_3\omega_3$. Thus,

$$\omega_3 \geq \frac{2}{I_3}\sqrt{MglI_1\cos\theta_0} \qquad \left(\theta_0 < \frac{\pi}{2}\right)$$

We therefore conclude that a steady precession can occur at the fixed angle of inclination θ_0 only if the angular velocity of spin is larger than the limiting value given by the above equation

From the equation for $\dot{\phi}$, i.e. $\dot{\phi} = \dfrac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$

we can write

$$\dot{\phi}_0 = \frac{\beta}{I_1 \sin^2\theta_0}$$

Hence we have two possible values for the precessional angular velocity $\dot{\phi}_0$, one for each β value.

If ω_3 (or p_ψ) is large then the square root in the expression for β can be expanded into the Taylor series, so that

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \left[1 - \frac{1}{2}\,\frac{4 Mgl I_1 \cos\theta_0}{p_\psi^2} + \dots\right]\right)$$

This gives

$$\dot{\phi}_0^{+} = \frac{\beta^{+}}{I_1 \sin^2\theta_0} = \frac{I_3 \omega_3}{I_1 \cos\theta_0} \quad \leftarrow \text{fast precession}$$

$$\dot{\phi}_0^{-} = \frac{\beta^{-}}{I_1 \sin^2\theta_0} = \frac{Mgl}{I_3 \omega_3} \quad \leftarrow \text{slow precession}$$

It is the slower of the two possible precessional angular velocities, $\dot{\phi}_0^{-}$, that is usually observed

Stability of free rigid body rotation

Let us consider a rigid body with principal moments of inertia such that $I_3 > I_2 > I_1$. We let the body axes to coincide with the principle axes.

Now let us assume that the rotation takes place mostly around the X-axis. In this case we can write

$$\vec{\omega} = \omega_1 \hat{e}_1 + \lambda \hat{e}_2 + \mu \hat{e}_3$$

where λ and μ are small compared with ω_1.

In the abscence of torques the Euler equations become

$$\begin{cases} (I_2 - I_3)\lambda\mu - I_1 \dot{\omega}_1 = 0 \\ (I_3 - I_1)\mu\omega_1 - I_2 \dot{\lambda} = 0 \\ (I_1 - I_2)\lambda\omega_1 - I_3 \dot{\mu} = 0 \end{cases}$$

$\mu\lambda$ is of the second order of smallness so we can neglect such a term. Then from the first eq. we set

$$\dot{\omega}_1 = 0 \Rightarrow \omega_1 = \text{const}$$

The other two equations yield

$$\dot{\lambda} = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\mu$$

$$\dot{\mu} = \left(\underbrace{\frac{I_1 - I_2}{I_3}\omega_1}_{\text{const}}\right)\lambda$$

The two coupled equations can be solved by differentiating one of them and substituting the first order derivative from anoter one:

$$\ddot{\lambda} = \left(\frac{I_3 - I_1}{I_2} \omega_1 \right) \dot{\mu}$$

$$\ddot{\lambda} + \left(\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3} \omega_1^2 \right) \lambda = 0$$

The solution is then

$$\lambda(t) = A e^{i \Omega_{1\lambda} t} + B e^{-i \Omega_{1\lambda} t} \quad \text{with} \quad \Omega_{1\lambda} = \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

Given our assumption, namely $I_3 > I_2 > I_1$, $\Omega_{1\lambda}$ happens to be real. Therefore $\lambda(t)$ is oscillatory and if it was small initially it will not grow beyond certain (small) magnitude with time. We can similarly investigate μ and obtain essentially the same result for it, $\Omega_{1\mu} = \Omega_{1\lambda} = \Omega_1$. Hence we can conclude that the small perturbations introduced to $\vec{\omega}$ that is along the x-axis do not increase with time. Consequently, such a rotation is stable.

If we carry out the same kind of analysis for

$$\vec{\omega} = \omega_3 \hat{e}_3 + \lambda \hat{e}_1 + \mu \hat{e}_2$$

i.e. for the case when the unperturbed $\vec{\omega}$ is along the z-axis, we will arrive at the same conclusion (Ω_3 can be obtained from Ω_1 by permuting $1 \leftrightarrows 2$:

$$\Omega_3 = \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

Again, Ω_3 is real and the motion is stable.

However for

$$\vec{\omega} = \omega_2 \hat{e}_2 + \lambda \hat{e}_1 + \mu \hat{e}_3$$

$$\Omega_2 = \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}}$$ is imaginary.

that means $\lambda(t)$ and $\mu(t)$ will not be oscillatory. They will grow exponentially and the motion is unstable.