General approach to a forced harmonic oscillator. Green's function

Suppose we have a forced harmonic oscillator

$$m\ddot{x} + kx = F(t) \qquad (*)$$

In the specific case when

$$F(t) = \begin{cases} F_0, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

i.e. when $F(t)$ was a step function. In this case it is easy to guess the particular solution, $x_p(t)$, of the non-homogenious equation. How do we obtain the solution if we are given some arbitrary (and nontrivial) $F(t)$? The feature of equation $(*)$ that we are going to exploit is its linearity. Suppose we have two particular solutions for $F_1(t)$ and $F_2(t)$

$$m\ddot{x}_1(t) + kx_1(t) = F_1(t)$$
$$m\ddot{x}_2(t) + kx_2(t) = F_2(t)$$

then

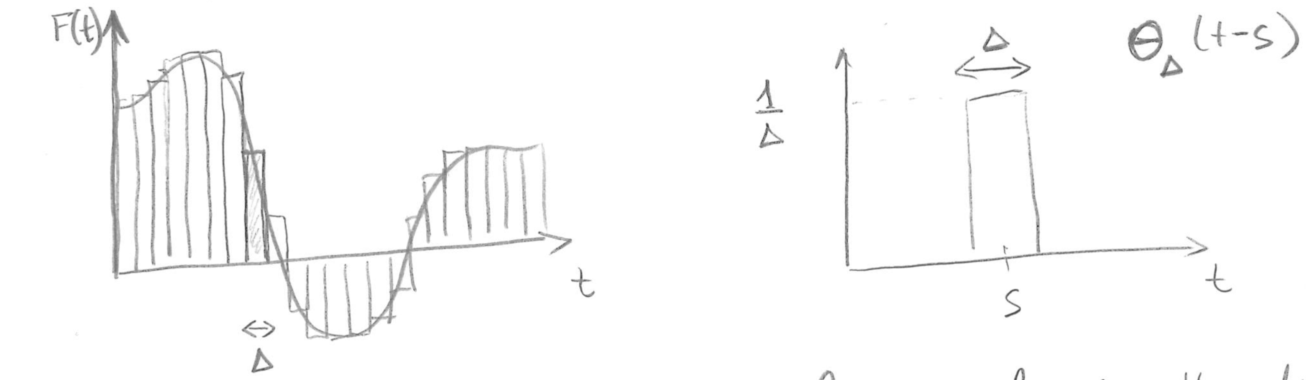
$$m\frac{d^2}{dt^2}\left[x_1(t) + x_2(t)\right] + k\left[x_1(t) + x_2(t)\right] = F_1(t) + F_2(t)$$

That is, we can find the solution to the problem with forcing function $F_1 + F_2$ if we knew the solutions to the problems with forcing functions F_1 and F_2.

This suggests that we could choose a simple set of forcing functions F, and solve the problem for these forcing functions. Then by adding the results with corresponding proportionality constants we can get the solution to the problem with an arbitrary $F(t)$.

There are three general ways that physicists commonly split the arbitrary force $F(t)$ and then reconstruct the solution. The first is the expansion into orthogonal basis sets. The Fourier series would be a typical example of this. The second way is integral transforms such as Laplace. The third way to find the particular solution for an arbitrary F is to use Green's functions.

The key idea of Green's function technique is illustrated in this figure:



We split the time into bins of small duration Δ centered at $t_n = n\Delta$ $\quad n = \ldots -2, -1, 0, 1, 2, \ldots$

If we define a square pulse of duration Δ and height $\frac{1}{\Delta}$, i.e.

$$\Theta_\Delta(s) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} \le s \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

then we can approximate $F(t)$ by $F_\Delta(t)$, defined as

$$F_\Delta(t) = \sum_n \overline{F}_n \, \Theta_\Delta(t - t_n) \cdot \Delta \qquad \text{where } \overline{F}_n = \frac{1}{\Delta} \int_{n\text{-th bin}} F(t)\, dt$$

We essentially represent $F(t)$ as a sum of square pulses. As Δ becomes smaller the approximation gets better

We can denote the particular solution of the nonhomogeneous equation with the forcing function $\Theta_\Delta(t)$ as $G_\Delta(t)$, i.e.

$$m\ddot{G}_\Delta + k G_\Delta = \Theta_\Delta$$

Then the peculiar solution with the forcing function $F_\Delta(t)$ is the following sum

$$X_\Delta(t) = \sum_n \bar{F}_n G_\Delta(t-t_n)\Delta$$

In the limit when $\Delta \to 0$ (and $F_\Delta \to F$) we get

$$X(t) = \int_{-\infty}^{+\infty} F(s)\, G(t-s)\, ds = \int_{-\infty}^{+\infty} F(t')\, G(t,t')\, dt'$$

$G(t,t')$ is called the Green function. Note that in the limit $\Delta \to 0$ $\Theta_\Delta(t-t') \to \delta(t-t')$ where $\delta(t)$ is the Dirac delta function.

In order to refresh memory let us recall the most important properties of the Dirac delta function:

$$\int_{-\infty}^{+\infty} \delta(t-t')\, dt' = 1 \qquad \int_{-\infty}^{+\infty} f(t)\,\delta(t-t')\, dt = f(t') \quad \forall f$$

$$\delta(t) = \frac{d}{dt}\eta(t)$$ $\eta(t)$ is the Heaviside step function

$\delta(t-t')$ is zero everywhere except the point $t=t'$, where it jumps to infinity (so that the integral of $\delta(t-t')$ is always a unity)

Now we need to find the particular solution of our equation for $F(t) = \delta(t-t')$. Let us do that. Consider

$$m\ddot{x} + kx = \delta(t)$$

We have to specify the initial conditions to solve for the Green function. Let us assume $x(t) = 0$, $t<0$

What will $x(t)$ be for $t>0$? Since there is no force after $t>0$ we have a free (i.e. homogenious) equation and its solution is

$$x = A \sin\omega t + B\cos\omega t \quad , \quad t>0$$

where A, B are determined by F that is applied at $t=0$.

Thus we need junction conditions that will connect the solution at $t<0$ (which is $x=0$) to the solution at $t>0$. Such conditions are found by looking at the equation for x

$$m\ddot{x} + kx = \delta(t)$$

$\leftarrow$ let us integrate both sides from $-\epsilon$ to $+\epsilon$ where ϵ is an infinitely small interval

$$m\int_{-\epsilon}^{+\epsilon} \ddot{x}(t)\,dt + k\int_{-\epsilon}^{+\epsilon} x(t)\,dt = \int_{-\epsilon}^{\epsilon} \delta(t)\,dt \Rightarrow m\dot{x}\Big|_{-\epsilon}^{\epsilon} = 1$$

The second terms on the left-hand side vanish because $x(t)$ is finite, but its second derivative $\ddot{x}(t)$ may be infinite at $t=0$.

$$m\dot{x}(t=0^+) - m\dot{x}(t=0^-) = 1$$

Since $\dot{x}(t=0^-) = 0$, we find $\dot{x}(t=0^+) = \frac{1}{m}$

We can now find A and B. Since $x(t=0^+) = x(t=0^-) = 0$

we have $B=0$. From the fact that

$\dot{x}(t=0^+)=\frac{1}{m}$ we set at $t=0$

$$A\omega=\frac{1}{m} \rightarrow A=\frac{1}{m\omega}$$

Thus $x(t)=\begin{cases}\frac{1}{m\omega}\sin\omega t, & t>0\\ 0, & t<0\end{cases}$

More generally, for forcing function $F(t)=\delta(t-t')$ we will have

$$x(t)=\begin{cases}\frac{1}{m\omega}\sin\omega(t-t'), & t>t'\\ 0, & t<t'\end{cases} \qquad G(t,t')=\begin{cases}\frac{1}{m\omega}\sin\omega(t-t') & t>t'\\ 0, & t<t'\end{cases}$$

We can now figure out what we should do in the case of an arbitrary forcing function $F(t)$. It can be represented as a bunch (essentially an infinite number) of delta functions

$$F(t)=\int_{-\infty}^{+\infty}F(t')\delta(t-t')\,dt'$$

Then we should write the particular solution for an arbitrary $F(t)$ as

$$x_p(t)=\int_{-\infty}^{+\infty}F(t')\,G(t,t')\,dt'$$

Suppose we want to find $x(t)$. Then we should take into account the effect of of all delta-functions at $t'<t$, but not $t'>t$ (causality)

Thus

$$x_p(t)=\int_{-\infty}^{t}F(t')\frac{1}{m\omega}\sin\omega(t-t')\,dt'$$