Oscillation of n point masses connected by a string (linear array of coupled harmonic oscillators)

Let us consider a model for a medium in 1D containing many particles bound by interparticle (interatomic) potential. When the particles do not get displaced by much from their equilibrium positions the interparticle potential can be approximated well by a quadratic function. Thus, such a system is essentially a linear array of coupled oscillators. This model may serve for describing a continuos medium, the propagation of waves through a continuous medium, or the vibrations of a crystalline lattice. We will consider a motion of a light elastic string fixed at both ends and loaded with n particles of mass m that are equally spaced along the string.

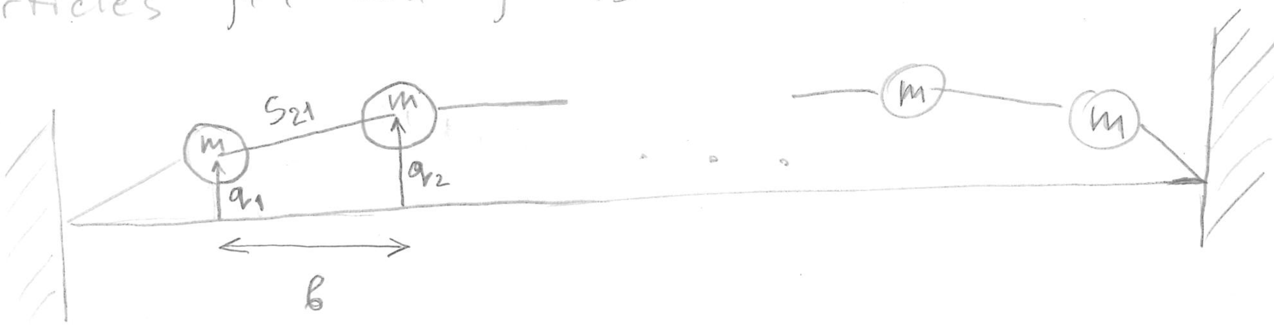
The displacements of those n particles will be labelled $q_1, \dots q_n$. Two types of displacements are possible: longitudinal and transverse. For simplicity we will assume that the motion is either longitudinal or transverse (although in an actual physical system a combination of the two occur)

The kinetic enegy is given by

$$T = \frac{m}{2}\left(\dot{q}_1^2 + \dots + \dot{q}_n^2\right)$$

In the longitudinal regime the stretch of the section of string between particle j and j+1 is $q_{j+1}-q_j$ and $\frac{1}{2}k\left(q_{j+1}-q_j\right)^2$ is the potential energy of that section of the string

In the transverse regime the distance between particles $j+1$ and j is



$$S_{j+1,j} = \sqrt{b^2 + (q_{j+1} - q_j)^2} = b\sqrt{1 + \frac{(q_{j+1}-q_j)^2}{b^2}} \approx b + \frac{(q_{j+1}-q_j)^2}{2b} + \dots$$

Hence the stretch is approximately $\frac{(q_{j+1}-q_j)^2}{2b}$ and the potential energy associated with this stretch is also of a quadratic form: $V_{j+1,j} = \tau \Delta S = \frac{\tau}{2b}(q_{j+1}-q_j)^2$, where τ is the tension force. The total potential energy is then

$$V = \frac{k}{2}\left[q_1^2 + (q_2-q_1)^2 + \dots + (q_n - q_{n-1})^2 + q_n^2\right]$$

where

$$k = \begin{cases} \frac{\tau}{b} = \frac{\text{tention in the string}}{\text{separatoon of particles}} & \leftarrow \text{transverse regime} \\ k = \text{elastis constant} & \leftarrow \text{longitudinal regime} \end{cases}$$

The Lagrangian of our system is

$$L = T - V = \frac{1}{2}\sum_{j=1}^{n}\left[m\dot{q}_j^2\right] - \frac{k}{2}\left[q_1^2 + \sum_{j=1}^{n-1}(q_{j+1}-q_j)^2 + q_n^2\right]$$

With this Lagranglan the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j}$$

are

$$m\ddot{q}_j = -k(q_j - q_{j-1}) + k(q_{j+1} - q_j) \qquad j = 1, \dots, n$$

To solve this system of equations we use the ansatz $q_j = \mathrm{Re}\left[a_j e^{i\omega t}\right]$ where a_j is the amplitude of vibration for the j-th particle. As a result of the substitution we get the following recursion formula

$$-m\omega^2 a_j = k\left(a_{j-1} - 2a_j + a_{j+1}\right) \qquad (*)$$

At the endpoints we set $a_0 = a_{n+1} = 0$

The secular equation for ω^2 is then

$$\begin{vmatrix} 2k - m\omega^2 & -k & 0 & \cdots & 0 \\ -k & 2k - m\omega^2 & -k & \cdots & 0 \\ 0 & -k & 2k - m\omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 2k - m\omega^2 \end{vmatrix} = 0$$

We can find the roots of the secular equation working with the simpler determinant:

$$D_n = \begin{vmatrix} \alpha & -1 & 0 & \cdots & 0 \\ -1 & \alpha & -1 & \cdots & 0 \\ 0 & -1 & \alpha & \cdots & 0 \\ \vdots & & & & \alpha \\ 0 & 0 & 0 & \cdots & \end{vmatrix} = 0 \qquad \text{where } \alpha = 2 - \frac{m\omega^2}{k}$$

Expanding the determinant of order n with respect to the first row we get

$$D_n = \alpha \underbrace{\left.\begin{vmatrix} \alpha & -1 & 0 & \cdots & 0 \\ -1 & \alpha & -1 & \cdots & 0 \\ 0 & -1 & \alpha & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \end{vmatrix}\right\}n-1}_{n-1} - (-1)\cdot \underbrace{\left.\begin{vmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & \alpha & -1 & 0 & \cdots & 0 \\ 0 & -1 & \alpha & -1 & \cdots & 0 \\ 0 & 0 & -1 & \alpha & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \alpha \end{vmatrix}\right\}n-1}_{n-1}$$

The first determinant is D_{n-1} while the second one is $(-1)\cdot D_{n-2}$, i.e.

$$D_n = \alpha D_{n-1} - D_{n-2} \qquad n \geq 2 \qquad (**)$$

Now it is easy to see that $D_1 = \alpha$ and

$$D_2 = \begin{vmatrix} \alpha & -1 \\ -1 & \alpha \end{vmatrix} = \alpha^2 - 1$$

and we can formally set $D_0 = 1$.

If we make a substitution in equation $(**)$ in the form

$$D_n = p^n$$

where p could, in principle be a function of n (although it will turn out to be not), then equation $(**)$ yields

$$p^n = \alpha p^{n-1} - p^{n-2}$$

or

$$p^2 - \alpha p + 1 = 0 \implies p = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

Now substituting $\alpha = 2\cos\delta$ we obtain for p

$$p = \cos\delta \pm \sqrt{\cos^2\delta - 1} = \cos\delta \pm i\sin\delta = e^{\pm i\delta}$$

Then

$$D_n = p^n = e^{\pm in\delta} = \cos n\delta \pm i\sin n\delta$$

Since equation $(**)$ is homogenious, the general solution is a linear combination

$$D_n = B\cos n\delta + C\sin n\delta$$

Since $D_0 = 1$ and $D_1 = \alpha = 2\cos\delta$, B and C are

$$B = 1 \qquad C = \cot\delta$$

and

$$D_n = \cos n\delta + \frac{\sin n\delta \cos\delta}{\sin\delta}$$

Using the relation

$$\sin\delta\cos n\delta + \sin n\delta\cos\delta = \sin(\delta + n\delta) = \sin(n+1)\delta$$

we obtain

$$D_n = \frac{\sin(n+1)\delta}{\sin\delta}$$

Now going back to the nontrivial solution of the secular equation we must have $D_n = 0$, or

$$\sin(n+1)\delta = 0 \quad \Rightarrow \quad \delta = \delta_s = \frac{s\pi}{n+1} \qquad s = 1, \dots, n$$

($s=0$ drops out since it leads to $\delta_0 = 0$ and hence to $D_n = n+1 \neq 0$)

Then we use $\alpha = D_1 = 2\cos\delta$, so

$$\alpha = 2 - \frac{m\omega^2}{k} = 2\cos\frac{s\pi}{n+1}$$

and ω is calculated from

$$\omega_s^2 = \frac{2k}{m}\left(1 - \cos\frac{s\pi}{n+1}\right) \qquad s = 1, \dots n$$

$$\omega_s = \sqrt{\frac{2k}{m}}\sqrt{1 - \cos\frac{s\pi}{n+1}}$$

These are the normal frequencies of the system.

The last expression allows simplification using the trigonometric identity $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$:

$$\omega_s = 2\omega_0 \sin\frac{s\pi}{2(n+1)} \qquad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

If we insert the previous expression for ω_s in the recursion formula (*) which reads $-m\omega^2 a_j = k(a_{j-1} - 2a_j + a_{j+1})$ then

$$-2\left(1 - \cos\frac{s\pi}{n+1}\right)a_j = a_{j-1} - 2a_j + a_{j+1}$$

or

$$a_{j-1} - 2a_j \cos\frac{s\pi}{n+1} + a_{j+1} = 0$$

$$2a_1 \cos\frac{s\pi}{n+1} = a_2 \qquad (a_0 = 0)$$

$$2a_n \cos\frac{s\pi}{n+1} = a_{n-1} \qquad (a_{n+1} = 0)$$

The system of equations for a_j is essentially the same as that for determinants D_n, with $\alpha = 2\cos\frac{s\pi}{n+1} = 2\cos\delta_n$, only the boundary conditions are different. The general solution for a_j is then

$$a_j = A'\cos j\delta_n + A\sin j\delta_n$$

$$= A'\cos\frac{s\pi j}{n+1} + A\sin\frac{s\pi j}{n+1}$$

Points $j=0$ and $j=n+1$ are tightly clamped, so that $a_0 = a_{n+1} = 0$. Then for $j=0$ we obtain

$$A' = 0 \quad \text{and} \quad a_j = A\sin\frac{s\pi j}{n+1}$$

this yield the following for q_j

$$q_j = A_s \sin\frac{s\pi j}{n+1}\cos\omega_s t$$

The general type of motion is a linear combination of all the normal modes:

$$q_j = \sum_{s=1}^{n} A_s \sin\left(\frac{s\pi j}{n+1}\right)\cos(\omega_s t - \phi_s)$$

or, alternatively,

$$q_j = \sum_{s=1}^{n} \sin\left(\frac{s\pi j}{n+1}\right)\left[\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t\right]$$

where constants A_s, ϕ_s (or β_s, γ_s) can be determined from the initial conditions:

$$q_j(0) = \sum_s \beta_s \sin\left(\frac{s\pi j}{n+1}\right) \qquad \dot{q}_j(0) = \sum_s \omega_s\gamma_s \sin\left(\frac{s\pi j}{n+1}\right)$$

Multiplying the first expression by $\sin\left(\frac{r\pi j}{n+1}\right)$ and summing over j we find

$$\sum_j{}' q_j(0) \sin\left(\frac{r\pi j}{n+1}\right) = \sum_{j,s}{}' \beta_s \sin\left(\frac{s\pi j}{n+1}\right) \sin\left(\frac{r\pi j}{n+1}\right)$$

It turns out that a trigonometric identity exist

$$\sum_{j=1}^{n}{}' \sin\left(\frac{s\pi j}{n+1}\right) \sin\left(\frac{r\pi j}{n+1}\right) = \frac{n+1}{2}\delta_{sr} \qquad s,r = 1,2,3,\dots$$

which allows to simplify the last expression:

$$\sum_j{}' q_j(0) \sin\left(\frac{r\pi j}{n+1}\right) = \sum_s{}' \beta_s \frac{n+1}{2}\delta_{sr} = \frac{n+1}{2}\beta_r$$

and we obtain the expression for β_r:

$$\beta_r = \frac{2}{n+1}\sum_j{}' q_j(0) \sin\left(\frac{r\pi j}{n+1}\right)$$

In a similar way we can obtain the expression for γ_r:

$$\gamma_r = \frac{2}{\omega_r(n+1)} \sum_j{}' \dot{q}_j(0) \sin\left(\frac{r\pi j}{n+1}\right)$$

Continuous string as a limiting case of a linear array of oscillators when $n \to \infty$

Now let us consider the case when the number of point masses approaches infinity, while the distance between them approaches zero, so that the linear mass density ρ is constant:

$$n \to \infty \quad , \quad b \to 0 \quad , \quad \frac{m}{b} = \rho = \text{const}$$

$$(n+1)b = L \quad \text{(total length of the string)}$$

then

$$\frac{s\pi j}{n+1} = s\pi \frac{jb}{(n+1)b} = s\pi \frac{x}{L}$$

where $x = jb$ specifies the position along the string of lenght L

$q_j(t)$ becomes a continuous function of a continuous variable x (and t) ;

$$q(x,t) = \sum_{s=1}^{\infty} \sin\left(\frac{s\pi x}{L}\right)\left[\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t\right]$$

Constants β_s and γ_s can be evaluated as follows

$$q(x,0) = \sum_{s=1}^{\infty} \beta_s \sin\left(\frac{s\pi x}{L}\right) \qquad \dot{q}(x,0) = \sum_{s=1}^{\infty} \omega_s \gamma_s \sin\left(\frac{s\pi x}{L}\right)$$

Eeach of these equations can be multiplied by $\sin\left(\frac{r\pi x}{L}\right)$ and integrated from $x=0$ to $x=L$, while making use of the relation

$$\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right)dx = \frac{L}{2}\delta_{rs}$$

so that

$$\beta_s = \frac{2}{L}\int_0^L q(x,0)\sin\left(\frac{s\pi x}{L}\right)dx$$

$$\gamma_s = \frac{2}{\omega_s L}\int_0^L \dot{q}(x,0)\sin\left(\frac{s\pi x}{L}\right)dx$$

The characteristic frequency ω_s can also be obtained in the limit $n\to\infty$ $\rho = \text{const}$:

$$\omega_s = 2\underbrace{\omega_0}_{\sqrt{\frac{k}{m}}}\sin\frac{s\pi}{2(n+1)} = 2\sqrt{\frac{\tau}{bm}}\sin\frac{s\pi}{2(n+1)} = \frac{2}{b}\sqrt{\frac{\tau}{\rho}}\sin\left(\frac{s\pi b}{2L}\right)$$

when $b\to 0$ we get

$$\omega_s = \frac{s\pi}{L}\sqrt{\frac{\tau}{\rho}} \qquad s = 1, 2, \ldots$$

In particular

$$\omega_1 = \frac{\pi}{L}\sqrt{\frac{\tau}{\rho}} \quad \text{— fundamental mode}$$