Phase velocity and dispersion

Let us consider a single "separated" solution to the wave equation $\frac{\partial^2 q}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 q}{\partial t^2}$ that is a product of two exponential factors:

$$q(x,t) = A e^{i\omega t} e^{-ikx} = A e^{i(\omega t - kx)} \qquad (*)$$

The reason we would like to focus our attention on this particular solution is that it has relevance to many practical situations (e.g. propagation of light wave in space). Moreover we can always represent a general solution as a linear combination of waves $(*)$ — a wave packet. If we know what happens to a single wave $(*)$ we can then find what happens to the wave packet. This is possible because of the linearity of the wave equation.

Solution $(*)$ is the 1D analogue of the so-called plane wave in 3D, $e^{i(\omega t - \vec{k}\cdot\vec{r})}$. The name plane wave originates from the fact that the value of the wave function $q(x,t)$ is the same for each set of points in space that lie in a plane (perpendicular to $\vec{k}$ vector).

In 1D the phase of the wave is defined as

$$\phi \equiv \omega t - kx$$

If we move our viewpoint along the x-axis at a velocity such that the phase at every point is the same, we see a stationary wave of the same shape. This velocity is called the phase velocity. It is the velocity of the propagation of the wave form.

If $\phi = \text{const}$ then $d\phi = 0$ and $d(\omega t - kx) = 0$

so $\omega\, dt = k\, dx$ and $v_p = \frac{dx}{dt} = \frac{\omega}{k} = v$

As we can see the phase velocity v_p is equal to the constant v that we had in the wave equation. It should be noted that the phase velocity is a well defined quantity only in the case when the shape of the wave function remains the same. Otherwise it is difficult to identify the instanteneous distance between two successive corresponding points of the wave. In other words, the wave length is constant (i.e. not function of time and position) only for a very extended (essentially infinite) wave.

In the case of a linear array of n coupled oscillators we had the following eigenfrequencies

$$\omega_s = 2\sqrt{\frac{\tau}{mb}}\sin\frac{s\pi}{2(n+1)} = \frac{2}{b}\sqrt{\frac{\tau}{\rho}}\sin\left(\frac{s\pi b}{2L}\right) = \frac{2}{b}\sqrt{\frac{\tau}{\rho}}\sin\left(\frac{\pi b}{\lambda_s}\right)$$

for $s=1$ there is no node between 0 and L (only at the ends), hence the wave length λ is equal to λ_1 $2L$ $\quad s=1$

$\lambda = L \quad s=2$

$\lambda = \frac{2L}{3} \quad s=3$

$\lambda = \frac{L}{2} \quad s=4$

$\vdots$

$\lambda_s = \frac{2L}{s}$

and

$$\omega_s = 2\sqrt{\frac{\tau}{mb}}\sin\left(\frac{k_s b}{2}\right) \qquad \text{where } k_s = \frac{2\pi}{\lambda_s} = \frac{s\pi b}{L}$$

Now if we consider the propagation of a small disturbance initiated at one end (zeroth particle)

$$q_0(t) = Ae^{i\omega t}$$

Any angular frequency smaller than $2\sqrt{\frac{\tau}{mb}}$ will be an allowed frequency. Then the phase velocity is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{\tau b}{m}} \frac{\sin \frac{kb}{2}}{\frac{kb}{2}} = v_p(k)$$

As one can see, the phase velocity is a function of the wave number. When $v_p = v_p(k)$ for a given medium, it is said to exhibit dispersion

In the limit of large wave length ($\lambda \to \infty$) or small k ($k \to 0$) the above expression becomes constant

$$v_p(\lambda \to \infty) = \sqrt{\frac{\tau b}{m}}$$

For the case of non-vanishing k, however, our medium (string) is dispersive. The limiting value of v_p corresponds to constant v in the wave equation.

In general, $v_p \neq v$