Lorentz transformation as a rotation through imaginary angle in 4D space

To illustrate the very peculiar interconnection between space and time in special relativity let us show that the Lorentz transformation equations connecting K and K' systems amount to a rotation about an imaginary angle in 4D space-time:

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

Lorentz transformation

If we introduce $x_1 = x \quad x_2 = y \quad x_3 = z \quad x_4 = ict$

and $x_1' = x' \quad x_2' = y' \quad x_3' = z' \quad x_4' = ict'$

then the above transformation looks as follows

$$\begin{cases} x_1' = \gamma(x_1 + i\beta x_4) \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = \gamma(x_4 - i\beta x_1) \end{cases} \qquad \beta \equiv \frac{v}{c}$$

Let us denote $\gamma = \cosh\alpha \quad \gamma\beta = \sinh\alpha$ (so $\beta = \tanh\alpha$)

We can do it because

$$\cosh^2\alpha - \sinh^2\alpha = 1$$

$$\gamma^2 - \gamma^2\beta^2 = \frac{1}{1-\beta^2} - \frac{1}{1-\beta^2}\beta^2 = 1$$

Then the transformation formula become

$$\begin{cases} x_1' = \cosh\alpha\, x_1 + i\sinh\alpha\, x_4 \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = -i\sinh\alpha\, x_1 + \cosh\alpha\, x_4 \end{cases}$$

Now $\cosh\alpha = \cos(i\alpha)$ and $i\sinh\alpha = \sin(i\alpha)$, so we can rewrite it as

$$\begin{cases} x_1' = \cos(i\alpha)\, x_1 + \sin(i\alpha)\, x_4 \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = -\sin(i\alpha)\, x_1 + \cos(i\alpha)\, x_4 \end{cases}$$

The rotation matrix in 4D space-time looks as follows

$$U(\alpha) = \begin{pmatrix} \cos i\alpha & 0 & 0 & \sin i\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin i\alpha & 0 & 0 & \cos i\alpha \end{pmatrix}$$

The inverse Lorentz transform, in turn, corresponds to the rotation about $-\alpha$:

$$U^{-1}(\alpha) = U(-\alpha) = \begin{pmatrix} \cos i\alpha & 0 & 0 & -\sin i\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin i\alpha & 0 & 0 & \cos i\alpha \end{pmatrix}$$

Length contraction

Let us consider a rod of length ℓ being at rest in frame K (laboratory frame). It turns out that if we measure the length of the same rod in the moving frame K' it will appear contracted. This phenomenon is called the Lorentz contraction.

Indeed if we assume that the rod lies along the x-axis in system K and system K' moves away from K with relative velocity v along x (and x') then an observer in K' measures the length in his/her own system by determining the differece in the coordinates of the ends of the rod, $x_2' - x_1'$. According to the Lorents transformation

$$x_1' = \gamma(x_1 - vt_1) \qquad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x_2' = \gamma(x_2 - vt_2)$$

so

$$x_2' - x_1' = \gamma\left(x_2 - x_1 - v(t_2 - t_1)\right)$$

In system K' where the observer measures the rod

$$t_2' = t_1'$$

therefore (from the Lorentz transformation for time)

$$\gamma\left(t_2 - \frac{vx_2}{c^2}\right) = \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$$

or

$$t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1)$$

Then length ℓ' as measured in K' is

$$\ell' \equiv x_2' - x_1' = \gamma\left(x_2 - x_1 - v(t_2 - t_1)\right) = \gamma\left(\ell - \frac{v^2}{c^2}\ell\right) = \sqrt{1-\frac{v^2}{c^2}}\,\ell = \frac{\ell}{\gamma} < \ell$$

Time dilation

Another interesting effect that follows from the Lorentz transformation is slowing down of moving clocks – time dilation.

Let us take a clock in the primed system, which is permanently attached at a certain position, say $x' = 0$ (we could place it elsewhere, too). According to the Lorentz transformation

$$x' = \gamma(x - vt)$$
$$0 = \gamma(x - vt)$$
$$x = vt$$

That is for the observer in the system K the clock moves with velocity v as expected. Looking at the expression for t' in the Lorentz transform we obtain

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = \gamma\left(t - \frac{v^2}{c^2}t\right) = \frac{t}{\gamma}$$

If we mark off equal time intervals on the t'-axis are the elapsed time intervals on the t-axis are shorter by factor γ, i.e.

$$\Delta t' = \frac{\Delta t}{\gamma} < \Delta t$$

The moving clock slows down, and this is called time dilation. Any process will be lenghten in time by factor γ. For example, a radioactive particle moving rapidly ($v \approx c$) will have longer lifetime than it would at rest.

We could reverse our setup and instead consider the clock fixed in system K, say at $x = 0$. Then

$$x = \gamma(x' + vt') \qquad\qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right) = \gamma\left(t' - \frac{v^2}{c^2}t\right) = \frac{t'}{\gamma}$$
and
$$x' = -vt'$$

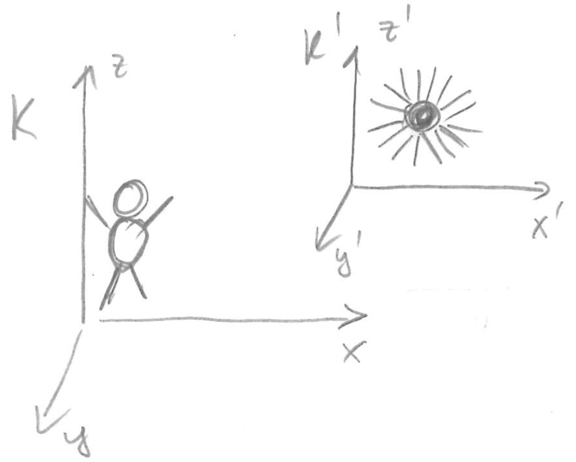
Again we arrive at the conclusion that the moving clock runs slower.

It is common to call the time measured on a clock fixed in a particular system the proper time (τ).

$$\Delta t' = \gamma \Delta \tau$$

Transformation law for frequency of light (Relativistic Doppler effect)

Let us consider how the frequency of electromagnetic waves changes when the source moves with some velocity v relative to the observer. Although the speed of light wave is the same for all observers (postulate 2), the color/frequency is not.



First let us consider the case when K' moves towards K.

In time interval Δt as measured by the observer the source emits signal whose spatial extent is m wave lengths. The total distance between the front and rear of the wave train is

$$\Delta X = c\Delta t - v \Delta t$$

The wave length is then

$$\lambda = \frac{\Delta X}{m} = \frac{(c-v)\Delta t}{m}$$

The frequency is given by

$$\nu = \frac{c}{\lambda} = \frac{cm}{(c-v)\Delta t}$$

If we place ourselves at the source we will determine that emits m wave lengths of frequency ν_0 in proper time $\Delta t'$:

$$m = \nu_0 \Delta t'$$

The proper time $\Delta t'$ in system K' is related to Δt in the observer's system K as (see previous topic)

$$\gamma \Delta t' = \Delta t \quad \text{or} \quad \Delta t' = \frac{\Delta t}{\gamma}$$

Hence

$$m = \frac{\nu_0 \Delta t}{\gamma}$$

When we substitute this into the expression for ν we get

$$\nu = \frac{cm}{(c-v)\Delta t} = \frac{c\nu_0}{\gamma(c-v)} = \frac{\nu_0}{\gamma}\frac{1}{1-\frac{v}{c}} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}\nu_0 =$$

$$= \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\,\nu_0 = \sqrt{\frac{1+\beta}{1-\beta}}\,\nu_0 > \nu_0 \quad \leftarrow \text{blue shift}$$

The case when the source moves away from the observer is similar – we just need to flip the sign of v:

$$\nu = \sqrt{\frac{1-\beta}{1+\beta}}\,\nu_0 < \nu_0 \quad \leftarrow \text{red shift}$$