StudentID: ____________________

PHYS 222 Classical Mechanics II (Spring 2019)
Instructor: Sergiy Bubin
Midterm Exam 1

Instructions:

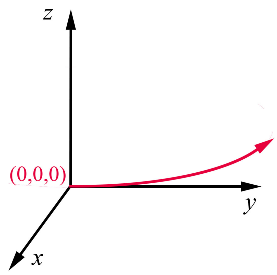
- All problems are worth the same number of points (although some might be more difficult than the others).
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that could be be useful may be provided in the appendix. Please look through it *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider the following functional:

$$F[y(x)] = \int_0^1 xyy'dx.$$

Evaluate the functional derivative $\frac{\delta F}{\delta y(x)}$.

Problem 2. From the Fermat principle it follows that for a medium with a nonuniform index of refraction the light may not necessarily propagate along a straight line. Now consider a medium with an index of refraction given by $n(x, y, z) = (1 + \alpha z)n_0$, where α and n_0 are constants. A narrow beam of light starts in this medium at point $(0, 0, 0)$ with the initial propagation vector along the y-direction. Find the function that describes the path of the beam in this medium.



Problem 3. Two particles with masses m_1 and m_2 are connected by a massless spring of unstretched length l and stiffness k. The system is free to rotate and vibrate on top of a frictionless horizontal plane that supports it.

(a) Find the Hamiltonian of the system.

(b) Write Hamilton's equations of motion.

(c) What generalized momenta, if any, are conserved?

Problem 4.

(a) A charged particle beam of circular cross section (radius R_0) is directed along the z-axis. The density of the particles across the beam volume is constant, but their momentum components transverse to the beam direction (p_x and p_y) are distributed uniformly over a circle of radius p_0 in the momentum space. Now, some focusing system reduces the beam radius from R_0 to R_1. Find the resulting distribution of the transverse momentum components.

(b) Consider a star of mass M and radius R composed of N atoms of mass m. Assuming that its interior density is uniform, find the temperature inside the star. You can recall the relation from the kinetic theory of gases that relates the average kinetic energy of an atom/molecule and temperature: $\overline{E}_{\text{kin}} = \frac{3}{2}kT$, where k is the Boltzmann constant.

Appendix: formula sheet

Lagrangian formalism

$$L = T - V, \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Least action principle

$$\delta S = 0, \qquad S = \int\limits_{t_1}^{t_2} L(q, \dot{q}, t)dt.$$

Beltrami identity

$$J[y] = \int\limits_a^b F(y, y')dx, \qquad F - y'\frac{\partial F}{\partial y'} = \text{const}.$$

Hamiltonian formalism

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i p_i \dot{q}_i - L, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \qquad \dot{f} = \frac{\partial f}{\partial t} + \{f, H\}.$$

Canonical transformation

$P = P(p, q),\ Q = Q(p, q)$ is canonical if leaves the form of the Hamilton equations unchanged. Also $\{Q, P\} = 1$.

Liouville's theorem

$$\sum_i \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) + \frac{\partial \rho}{\partial t} = 0.$$

Virial theorem

$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$. For a system with the interaction potential $V(r) = \alpha r^n$ we have $\overline{T} = \frac{n}{2}\overline{V}$

Useful integrals

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right) + C$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arctan\left[\frac{x}{\sqrt{a^2 - x^2}}\right]\right) + C$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C = \operatorname{arccosh}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C, \quad a \neq 0$$