1 If $\delta y(x) = \varepsilon \eta(x)$ (ε is an infinitesimal parameter)

the variation of F is

$$\delta F = F[y+\varepsilon\eta] - F[y] = \int_0^1 x(y+\varepsilon\eta)(y'+\varepsilon\eta')\,dx - \int_0^1 xyy'\,dx =$$

$$= \varepsilon \int_0^1 xy\eta'\,dx + \varepsilon\int_0^1 xy'\eta\,dx + O(\varepsilon^2)$$

The first integral can be manipulated by doing the integration by parts

$$\int_0^1 xy\frac{d\eta}{dx}\,dx = \underbrace{xy\eta\Big|_0^1}_{\text{vanishes, because } \eta(0)=\eta(1)=0} - \int_0^1 \eta \frac{d}{dx}(xy) = -\int_0^1 (y+xy')\eta\,dx$$

With that we have

$$\delta F = \varepsilon \int_0^1 (-y - xy' + xy')\eta\,dx + O(\varepsilon^2) = \varepsilon\int_0^1 \underbrace{(-y)}_{\frac{\delta F}{\delta y}}\eta\,dx$$

Therefore,

$$\frac{\delta F}{\delta y(x)} = -y(x)$$

② Since the light starts propagating along the y-axis and n is a function of the z coordinate only, the path will lie in the yz plane. So we can ignore the x coordinate completely. The total travel time is given by the following integral over the path of light:

$$T = \int \frac{ds}{v}$$

Here $ds = \sqrt{1+z_y'^2}\, dy$ $\quad \frac{1}{v} = \frac{n}{c} = (1+\alpha z)\frac{n_0}{c}$. The path $z(y)$ has to minimize the functional

$$T = \frac{n_0}{c}\int \sqrt{1+z_y'^2}\,(1+\alpha z)\, dy \qquad \text{with } F(z,z',y) = \sqrt{1+z'^2}(1+\alpha z)$$

The integrand, F, does not depend on y. Therefore we can use the Beltrami identity as a condition of an extremum:

$$F - z'\frac{\partial F}{\partial z'} = b = \text{const}.$$

In our case this gives

$$\sqrt{1+z'^2}\,(1+\alpha z) - \frac{z'^2}{\sqrt{1+z'^2}}(1+\alpha z) = b$$

$$\left(\sqrt{1+z'^2} - \frac{z'^2}{\sqrt{1+z'^2}}\right)^2 (1+\alpha z)^2 = b^2$$

$$\left(1+z'^2 + \frac{z'^4}{1+z'^2} - 2z'^2\right)^2 (1+\alpha z)^2 = b^2$$

$$\frac{(1+\alpha z)^2}{1+z'^2} = b^2 \implies z'^2 = \frac{(1+\alpha z)^2}{b^2} - 1$$

$$z' = \sqrt{\frac{(1+\alpha z)^2}{b^2} - 1}$$

$$\int dy = \int \frac{dz}{\sqrt{\frac{(1+\alpha z)^2}{b^2} - 1}} = \frac{b^2}{\alpha}\int \frac{dz}{\sqrt{\left(z+\frac{1}{\alpha}\right)^2 - \frac{b^2}{\alpha^2}}}$$

If we make a substitution $\frac{1}{\gamma}\left(z+\frac{1}{\alpha}\right) = u$ $\quad \frac{1}{\gamma}dz = du$ $\quad \gamma = \frac{b}{\alpha}$

we will get

$$\int dy = \frac{b^2}{\alpha} \int \frac{du}{\sqrt{u^2-1}}$$

or

$$y + k = \frac{b^2}{\alpha} \operatorname{arccosh}(u) = \frac{b^2}{\alpha} \operatorname{arccosh}\left(\frac{z+\frac{1}{\alpha}}{\frac{b}{\alpha}}\right) = \frac{b^2}{\alpha} \operatorname{arccosh}\left(\frac{1+\alpha z}{b}\right)$$

where k is a constant

Solving for z as a function of y gives

$$z = \frac{1}{\alpha}\left[b \cosh\left(\frac{\alpha}{b^2}(y+k)\right) - 1\right]$$

Constants b and k can be determined using the condition $z(y=0)=0$ and $z'(y=0)=0$:

$$b \cosh\left(\frac{\alpha k}{b^2}\right) - 1 = 0 \quad \text{and} \quad \frac{1}{b} \sinh\left(\frac{\alpha k}{b^2}\right) = 0$$

which yields

$$k = 0 \qquad b = 1$$

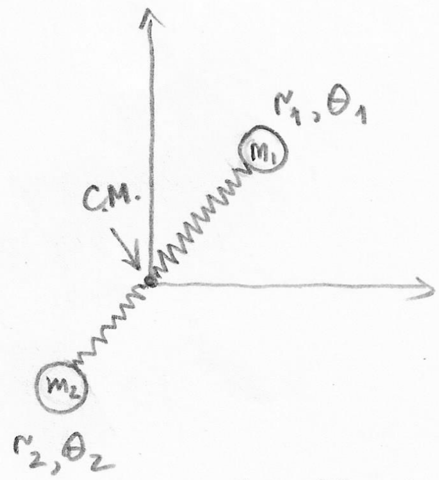
With that the final expression for the trajectory is

$$z = \frac{1}{\alpha}\left[\cosh(\alpha y) - 1\right]$$

or

$$y = \frac{1}{\alpha} \operatorname{arccosh}(1 + \alpha z)$$

③



Let us place the coordinate system at the center of mass. The position of each particle with respect to to origin can then be defined by the distance r_i and angle θ_i

The Lagrangian of the system is

$$L = T - V = \frac{m_1}{2}\left(\dot{r}_1^2 + r_1^2 \dot{\theta}_1^2\right) + \frac{m_2}{2}\left(\dot{r}_2^2 + r_2^2 \dot{\theta}_2^2\right) - \frac{k}{2}\left(r_1 + r_2 - \ell\right)^2$$

a) Let us determine the Hamiltonian:

$$p_{1r} = \frac{\partial L}{\partial \dot{r}_1} = m\dot{r}_1 \qquad\qquad \dot{r}_1 = \frac{p_{1r}}{m_1}$$

$$p_{1\theta} = \frac{\partial L}{\partial \dot{\theta}_1} = m r_1^2 \dot{\theta}_1 \qquad \Longrightarrow \qquad \dot{\theta}_1 = \frac{p_{1\theta}}{m_1 r_1^2}$$

$$p_{2r} = \frac{\partial L}{\partial \dot{r}_2} = m\dot{r}_2 \qquad\qquad \dot{r}_2 = \frac{p_{2r}}{m_2}$$

$$p_{2\theta} = \frac{\partial L}{\partial \dot{\theta}_2} = m \dot{r}_2^2 \dot{\theta}_2 \qquad\qquad \dot{\theta}_2 = \frac{p_{2\theta}}{m_2 r_2^2}$$

$$H = \sum_i p_i \dot{q}_i - L = \frac{p_{1r}^2}{m_1} + \frac{p_{1\theta}^2}{m_1 r_1^2} + \frac{p_{2r}^2}{m_2} + \frac{p_{2\theta}^2}{m_2 r_2^2} - \frac{m_1}{2}\left(\frac{p_{1r}^2}{m_1^2} + \frac{p_{1\theta}^2}{m_1^2 r_1^2}\right) -$$

$$- \frac{m_2}{2}\left(\frac{p_{2r}^2}{m_2^2} + \frac{p_{2\theta}^2}{m_2^2 r_2^2}\right) + \frac{k}{2}\left(r_1 + r_2 - \ell\right)^2 = \frac{p_{1r}^2}{2m_1} + \frac{p_{1\theta}^2}{2m_1 r_1^2} + \frac{p_{2r}^2}{2m_2} + \frac{p_{2\theta}^2}{2m_2 r_2^2} + \frac{k}{2}\left(r_1 + r_2 - \ell\right)^2.$$

b) Hamilton's equations of motion:

$$\dot{r}_1 = \frac{\partial H}{\partial p_{1r}} \qquad \dot{p}_{1r} = -\frac{\partial H}{\partial r_1} \qquad\qquad \dot{r}_1 = \frac{p_{1r}}{m_1} \qquad\qquad \dot{p}_{1r} = -k\left(r_1 + r_2 + \ell\right) + \frac{p_{1\theta}^2}{m_1 r_1^3}$$

$$\dot{\theta}_1 = \frac{\partial H}{\partial p_{1\theta}} \qquad \dot{p}_{1\theta} = -\frac{\partial H}{\partial \theta_1} \qquad\qquad \dot{\theta}_1 = \frac{p_{1\theta}}{m_1 r_1^2} \qquad\qquad \dot{p}_{1\theta} = 0$$

$$\Longrightarrow$$

$$\dot{r}_2 = \frac{\partial H}{\partial p_{2r}} \qquad \dot{p}_{2r} = -\frac{\partial H}{\partial r_2} \qquad\qquad \dot{r}_2 = \frac{p_{2r}}{m_2} \qquad\qquad \dot{p}_{2r} = -k\left(r_1 + r_2 + \ell\right) + \frac{p_{2\theta}^2}{m_2 r_2^3}$$

$$\dot{\theta}_2 = \frac{\partial H}{\partial p_{2\theta}} \qquad \dot{p}_{2\theta} = -\frac{\partial H}{\partial \theta_2} \qquad\qquad \dot{\theta}_2 = \frac{p_{2\theta}}{m_2 r_2^2} \qquad\qquad \dot{p}_{2\theta} = 0$$

these recover the definition of our generalized momenta

c) From the Hamilton's equations it follows that

$$p_{1\theta} = const$$

$$p_{2\theta} = const$$

(4) a) Here we apply the Liouville theorem, which states that the phase space volume should be preserved:

$$\underbrace{(\pi R^2)}_{V_r} \cdot \underbrace{(\pi p^2)}_{V_{p_r}} = \text{const}$$

So

$$R_0^2 p_0^2 = R_1^2 p_1^2 \quad \text{and} \quad p_1 = \frac{R_0}{R_1} p_0$$

b) As we know it from the introductory physics the gravitational potential energy of a spherical object is given by

$$V(R) = -\int_0^R G \frac{\left(\frac{4}{3}\pi \rho r^3\right)\left(4\pi r^2 \rho\, dr\right)}{r} = -\frac{16}{3}\pi^2 G \rho^2 \int_0^R r^4 dr = -\frac{16}{15}\pi^2 \rho^2 G R^5$$

or using the fact that $M = \frac{4}{3}\pi R^3 \rho$

$$V(R) = -\frac{3GM^2}{5R}$$

For a system with interparticle interaction $\propto \frac{\alpha}{r_{ij}}$ the virial theorem gives

$$\overline{E}_{kin} = -\frac{1}{2}\overline{V}$$

or

$$N \cdot \frac{3}{2} kT = -\frac{1}{2}\left(-\frac{3GM^2}{5R}\right)$$

From here we determine that

$$T = \frac{1}{5}\frac{GM^2}{kNR}$$

Using the fact that $m = \frac{M}{N}$ we could also rewrite it as

$$T = \frac{1}{5}\frac{GmM}{kR}$$