StudentID: ______________________

PHYS 222 Classical Mechanics II (Spring 2019)
Instructor: Sergiy Bubin
Midterm Exam 2

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others).
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that could be be useful may be provided in the appendix. Please look through it *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

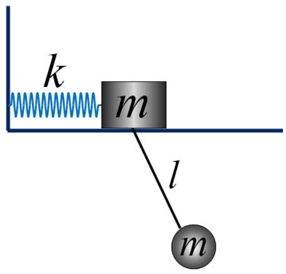
Problem 1. A uniform rectangular block of mass m and dimensions a, $2a$, and $3a$ spins about an axis that passes through its center and one of the corners with angular velocity ω.

(a) What is the kinetic energy of the block?

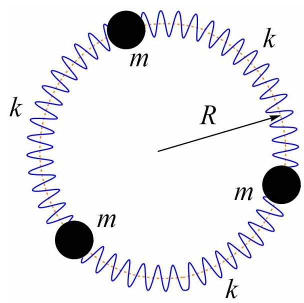
(b) What is the angle between the angular velocity vector and the angular momentum vector?

Problem 2. Consider a thin lamina (plate) of an arbitrary shape. Suppose the lamina is in xy-plane (body frame) and rotates freely under zero external torque. Use the Euler equations to show that the projection of $\boldsymbol{\omega}$ on the plane of the lamina remains constant in time, i.e. $\omega_x^2 + \omega_y^2 = \text{const}$. For what kind of lamina ω_z remains constant as well?

Problem 3. A pendulum of mass m and length l is suspended from a block, also of mass m. The block, in turn, is attached to a wall with a spring of stiffness k, as shown in the figure. The block moves on the horizontal surface without friction. Find the normal frequencies of this system.



Problem 4. The system shown in the figure consists of three small identical beads of mass m. The beads are connected with three springs of stiffness k and wind around a ring or radius R. The equilibrium length of each spring is $\frac{2\pi}{3}R$. Find the normal frequencies and normal modes of the system. Give physical interpretation of those frequencies and modes.



Appendix: formula sheet

Lagrangian formalism

$$L = T - V, \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Least action principle

$$\delta S = 0, \qquad S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt.$$

Beltrami identity

$$J[y] = \int_a^b F(y, y')dx, \qquad F - y' \frac{\partial F}{\partial y'} = \text{const.}$$

Hamiltonian formalism

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i p_i \dot{q}_i - L, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \qquad \dot{f} = \frac{\partial f}{\partial t} + \{f, H\}.$$

Canonical transformation

$P = P(p, q),\ Q = Q(p, q)$ is canonical if leaves the form of the Hamilton equations unchanged. Also $\{Q, P\} = 1$.

Liouville's theorem

$$\sum_i \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) + \frac{\partial \rho}{\partial t} = 0.$$

Virial theorem

$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}.$ For a system with the interaction potential $V(r) = \alpha r^n$ we have $\overline{T} = \frac{n}{2}\overline{V}$

Definition of the tensor of inertia

$$I = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix}$$

Displaced axis theorem

Tensor of inertia about an origin displaced by a constant vector $\mathbf{a}$ is given by
$(I_{\mathbf{a}})_{\alpha\beta} = (I_{\text{c.m.}})_{\alpha\beta} + M(a^2\delta_{\alpha\beta} - a_\alpha a_\beta)$

Moment of inertia about an axis defined by a normal vector

Moment of inertia $I_{\mathbf{n}}$ about an axis that passes through the center of mass and is defined by a normal vector $\mathbf{n}$ is given by (here I is the tensor of inertia of the system):
$I_{\mathbf{n}} = \mathbf{n}^T I \mathbf{n}$

Euler angles and the transformation between the lab and body frame

If $\mathbf{r}'$ is the position in the fixed/lab frame and $\mathbf{r}$ is the position in the body frame then $\mathbf{r} = U_\psi U_\theta U_\phi \mathbf{r}' = U\mathbf{r}'$, where

$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad U_\phi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler equations for a rigid body

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

Normal frequencies and normal modes of a system of n coupled harmonic oscillators

Generalized eigenvalue problem: $K\mathbf{a}^{(i)} = \omega_i^2 M\mathbf{a}^{(i)}$
Trajectories: $\mathbf{x}(t) = \sum_i \mathbf{a}^{(i)}\mathrm{Re}[e^{i\omega_i t}]$

Useful integrals

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right) + C$$

$$\int \sqrt{a^2 - x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arctan\left[\tfrac{x}{\sqrt{a^2-x^2}}\right]\right) + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C = \mathrm{arccosh}\,\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad a \neq 0$$