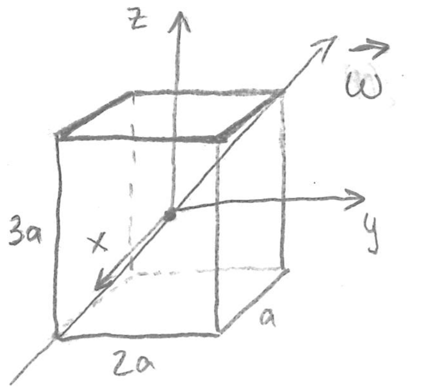
(1) First, let us determine the principal moments of inertia of our block ($m = 6\rho a^3$)



$$I_1 = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-a}^{a} \int_{-\frac{3a}{2}}^{\frac{3a}{2}} (y^2+z^2)\, dx\, dy\, dz =$$

$$= \rho\left(a \cdot \frac{y^3}{3}\Big|_{-a}^{a} \cdot 3a + a \cdot 2a \cdot \frac{z^3}{3}\Big|_{-\frac{3a}{2}}^{\frac{3a}{2}}\right) = \rho\left(2a^5 + \frac{9}{2}a^5\right) = \frac{13}{2}\rho a^5 = \frac{13}{12} m a^2$$

$$I_2 = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-a}^{a} \int_{-\frac{3a}{2}}^{\frac{3a}{2}} (x^2+z^2)\, dx\, dy\, dz = \rho\left(\frac{x^3}{3}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} \cdot 2a \cdot 3a + a \cdot 2a \cdot \frac{z^3}{3}\Big|_{-\frac{3a}{2}}^{\frac{3a}{2}}\right) =$$

$$= \rho\left(\frac{1}{2}a^5 + \frac{9}{2}a^5\right) = \frac{10}{2}\rho a^5 = \frac{10}{12} m a^2$$

$$I_3 = \rho \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-a}^{a} \int_{-\frac{3a}{2}}^{\frac{3a}{2}} (x^2+y^2) = \rho\left(\frac{1}{2}a^5 + 2a^5\right) = \frac{5}{2}\rho a^5 = \frac{5}{12} m a^2$$

In the coordinate frame chosen in the sketch the tensor of inertia is diagonal.

a) The kinetic energy is $T = \frac{1}{2}\vec{\omega}^T I \vec{\omega}$

$$\vec{\omega} = \frac{\omega}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \text{so} \quad T = \frac{\omega^2 m a^2}{14}(1\ 2\ 3)\begin{pmatrix}\frac{13}{12} & 0 & 0\\ 0 & \frac{10}{12} & 0\\ 0 & 0 & \frac{5}{12}\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{7}{24} m a^2 \omega^2$$

b) $$\vec{L} = I\vec{\omega} = m a^2 \begin{pmatrix}\frac{13}{12} & 0 & 0\\ 0 & \frac{10}{12} & 0\\ 0 & 0 & \frac{5}{12}\end{pmatrix} \frac{\omega}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{m a^2 \omega}{12\sqrt{14}}\begin{pmatrix}13\\20\\15\end{pmatrix}$$

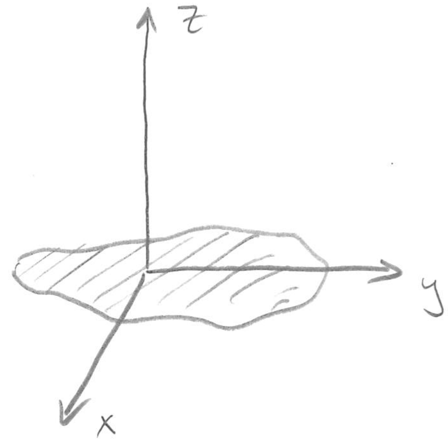
The cosine of the angle α between $\vec{\omega}$ an $\vec{L}$ is:

$$\cos\alpha = \frac{\vec{\omega}\cdot\vec{L}}{|\vec{\omega}||\vec{L}|} = \frac{2T}{|\vec{\omega}||\vec{L}|} = \frac{\frac{7}{12} m a^2 \omega^2}{\omega \cdot \frac{m a^2 \omega}{12\sqrt{14}}\sqrt{13^2+20^2+15^2}} =$$

$$= \frac{7\sqrt{7}}{\sqrt{397}} \approx 0.9295$$

$$\alpha = \arccos\left(\frac{7\sqrt{7}}{\sqrt{397}}\right)$$

(2) The tensor of inertia of a thin lamina (as shown in the sketch) is



$$I = \begin{pmatrix} \sum_i m_i y_i^2 & -\sum_i m_i x_i y_i & 0 \\ -\sum_i m_i x_i y_i & \sum_i m_i x_i^2 & 0 \\ 0 & 0 & \sum_i m_i (x_i^2 + y_i^2) \end{pmatrix}$$

where the sum (or integral) is over all particles that the lamina consists of. It is obvious that this tensor can be diagonalized by a certain rotation in the xy-plane. Moreover, the trace of the upper 2×2 submatrix under such rotation (i.e. $I_{11} + I_{22}$) is going to be constant as we know that the trace of a matrix is invariant under rotations. Therefore, when we diagonalize I, it will have the form

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

where $I_1 + I_2 = I_3$. This is because $I_{11} + I_{22} = I_{33}$ before the diagonalization.

Now let us consider the Euler equations for a torque-free motion:

$$I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) = 0$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) = 0$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0$$

Given what we know that $I_1 + I_2 = I_3$ they can be rewritten as

$$I_1 \dot{\omega}_1 + \omega_2 \omega_3 I_1 = 0$$

$$I_2 \dot{\omega}_2 - \omega_3 \omega_1 I_2 = 0$$

$$I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) = 0$$

We can multiply the first two of them by $\frac{\omega_1}{I_1}$ and $\frac{\omega_2}{I_2}$ respectively and get

$$\omega_1 \dot{\omega}_1 + \omega_1 \omega_2 \omega_3 = 0$$

$$\omega_2 \dot{\omega}_2 - \omega_1 \omega_2 \omega_3 = 0$$

After adding them together we get

$$\omega_1 \dot{\omega}_1 + \omega_2 \dot{\omega}_2 = 0 \qquad \text{or} \qquad \frac{1}{2} \frac{d}{dt} \left(\omega_1^2 + \omega_2^2 \right) = 0$$

This proves that

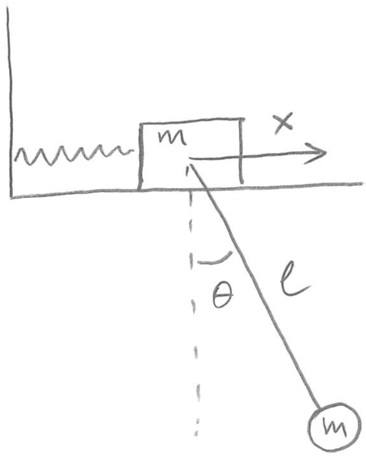
$$\omega_1^2 + \omega_2^2 = \text{const}.$$

If we take a look at the trird Euler equation we can see that if $I_1 = I_2$ then

$$\dot{\omega}_3 = 0 \qquad \text{and} \qquad \omega_3 = \text{const}$$

Therefore the condition for $\omega_3 \equiv \omega_z$ to remain constant is $I_1 = I_2$ (symmetric top)

③ First, let us write the Lagrangian of the system. The kinetic energy is:



$$T = \frac{1}{2} m \left[\dot{x}^2 + \left(\frac{d}{dt}(x + \ell \sin\theta) \right)^2 + \right.$$
$$\left. + \left(\frac{d}{dt} \ell \cos\theta \right)^2 \right] = m \left[\dot{x}^2 + \frac{1}{2} \ell^2 \dot{\theta}^2 + \ell \dot{x} \dot{\theta} \cos\theta \right]$$

The potential energy is

$$V = \frac{kx^2}{2} - mg\ell\cos\theta$$

when θ is small the Lagrangian becomes $\left(\theta \approx 1 - \frac{\theta^2}{2}\right)$

$$L = m\left(\dot{x}^2 + \frac{1}{2}\ell^2\dot{\theta}^2 + \ell\dot{x}\dot{\theta} \right) - \frac{kx^2}{2} - mg\ell\frac{\theta^2}{2}$$

It is convenient to denote $q_1 = x$ $\quad q_2 = \ell\theta$ $\quad \vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

$$L = m\left(\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \dot{q}_1\dot{q}_2 \right) - \frac{kq_1^2}{2} - \frac{mg}{\ell}\frac{q_2^2}{2}$$

The equations of motion for the system are

$$2m\ddot{q}_1 + m\ddot{q}_2 = -kq_1$$

$$m\ddot{q}_2 + m\ddot{q}_1 = -\frac{mg}{\ell} q_2$$

which we can divide by m and write in matrix form

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \ddot{\vec{q}} = - \underbrace{\begin{pmatrix} \frac{k}{m} & 0 \\ 0 & \frac{g}{\ell} \end{pmatrix}}_{B} \vec{q}$$

The normal frequencies are found by solving the secular equation $\det(B - \omega^2 A) = 0$

$$\begin{vmatrix} \frac{k}{m} - 2\omega^2 & -\omega^2 \\ -\omega^2 & \frac{g}{\ell} - \omega^2 \end{vmatrix} = 0 \implies \left(\frac{k}{m} - 2\omega^2\right)\left(\frac{g}{\ell} - \omega^2\right) - \omega^4 = 0$$

$$\omega_{1,2}^2 = \frac{1}{2}\left(\frac{k}{m} + 2\frac{g}{\ell} \pm \sqrt{\frac{k^2}{m^2} - 4\frac{g^2}{\ell^2}} \right)$$

$$\omega_{1,2} = \frac{1}{\sqrt{2}} \sqrt{ \frac{k}{m} + 2\frac{g}{\ell} \pm \sqrt{\frac{k^2}{m^2} - 4\frac{g^2}{\ell^2}} }$$

(4) We will use angles θ_1, θ_2, and θ_2 to define the positions of the beads on the ring. The Lagrangian of the system is:

$$L = \frac{1}{2} m R^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) + \frac{1}{2} k R^2 \left[\left(\theta_2 - \theta_1 - \frac{2\pi}{3}\right)^2 + \left(\theta_3 - \theta_2 - \frac{2\pi}{3}\right)^2 + \right.$$

$$\left. + \left(\theta_1 - \theta_3 - \frac{2\pi}{3}\right)^2 \right] =$$

$$= \frac{1}{2} m R^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) + \frac{1}{2} k R^2 \left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + (\theta_1 - \theta_3)^2 \right]$$

The equations of motion are (when divided by R^2):

$$\begin{aligned} m\ddot{\theta}_1 - k(\theta_2 - \theta_1) + k(\theta_1 - \theta_3) &= 0 \\ m\ddot{\theta}_2 + k(\theta_2 - \theta_1) - k(\theta_3 - \theta_2) &= 0 \\ m\ddot{\theta}_3 + k(\theta_3 - \theta_2) - k(\theta_1 - \theta_3) &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} m\ddot{\theta}_1 &= -k(2\theta_1 - \theta_2 - \theta_3) \\ m\ddot{\theta}_2 &= -k(-\theta_1 + 2\theta_2 - \theta_3) \\ m\ddot{\theta}_3 &= -k(-\theta_1 - \theta_2 + 2\theta_3) \end{aligned}$$

or, in matrix form:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{pmatrix} = -\omega_0^2 \underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}}_{\mathbb{K}} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \qquad \text{where } \omega_0^2 \equiv \frac{k}{m}$$

For oscillatory motion we seek the solution of this system of equations as

$$\vec{\theta} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i\omega t} = \vec{a} e^{i\omega t}$$

which yields the generalized eigenvalue problem

$$\mathbb{K} \vec{a} = \omega^2 M \vec{a}$$

The eigenvalues are obtained by solving $\det(K - \omega^2 M) = 0$:

where $\lambda \equiv \frac{\omega^2}{\omega_0^2}$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)^3 - 2 - 3(2-\lambda) = 0$$

$\lambda_1 = 3 \qquad \lambda_2 = 3 \qquad \lambda_3 = 0$

i.e. $\omega_1 = 3\omega_0 \qquad \omega_2 = 3\omega_0 \qquad \omega_3 = 0$

The corresponding eigenvectors $\vec{a}^{(i)}$ can be easily computed (I will skip this basic algebra here) and they are

$$\vec{a}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{a}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \vec{a}^{(3)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The physical interpretation of the the modes is as follows:

(1) bead 2 is at rest, the other two beads oscillate in opposite directions about bead 2

(2) bead 3 is at rest, the other two beads oscillate in opposite directions about bead 3

(3) all beads undergo rotational motion together (effectively there is no oscillation)