StudentID: ____________________

PHYS 222 Classical Mechanics II (Spring 2019)
Instructor: Sergiy Bubin
Midterm Exam 3

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others).
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that could be be useful may be provided in the appendix. Please look through it *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. In lecture we used the method of separation of variables to obtain the general solution $q(x,t)$ to the problem of a vibrating continuos string fixed at both ends, that is $q(0,t) = 0$ and $q(L,t) = 0$ (the Dirichlet boundary conditions), given the initial conditions $q(x,0) = f(x)$ and $\dot{q}(x,0) = g(x)$. Use the same method to solve a related problem, in which everything is the same except that the ends of the string are loose (e.g. you can imagine that the ends have small rings that can slide up and down on two thin frictionless vertical columns). Mathematically these boundary conditions (called the Neuman boundary conditions) are stated as $\frac{\partial q}{\partial x}\big|_{x=0} = 0$ and $\frac{\partial q}{\partial x}\big|_{x=L} = 0$.

Problem 2. The dispersion relation for waves propagating on the surface of water in the presence of gravity force is known to be $\omega^2 = gk\tanh(kh)$, where g is the acceleration by gravity and h is the water depth. Answer the following questions and be specific when you do so (i.e. provide specific expresions whenever possible):

(a) Is there any difference between phase velocity and group velocity for water waves? How do they compare in the open ocean?

(b) Do waves get smaller or taller as they approach a beach/shore? Do waves break at a beach?

Problem 3. Consider the propagation of light in a moving medium (e.g. water). The index of refraction of the medium is n. The medium moves with velocity v (not negligible compared to the speed of light c) away from the origin in the laboratory frame K.

(a) What is the speed of light propagating in the medium as measured in the laboratory frame K?

(b) How does it differ from the speed of light when the medium is stationary?

(c) If the velocity of the medium is $v = c$, what does the answer to question (a) becomes?

Appendix: formula sheet

Lagrangian formalism

$$L = T - V, \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0.$$

Least action principle

$$\delta S = 0, \qquad S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt.$$

Beltrami identity

$$J[y] = \int_a^b F(y, y')dx, \qquad F - y'\frac{\partial F}{\partial y'} = \text{const.}$$

Hamiltonian formalism

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i p_i \dot{q}_i - L, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Poisson bracket

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \qquad \dot{f} = \frac{\partial f}{\partial t} + \{f, H\}.$$

Canonical transformation

$P = P(p, q),\ Q = Q(p, q)$ is canonical if leaves the form of the Hamilton equations unchanged. Also $\{Q, P\} = 1$.

Liouville's theorem

$$\sum_i \left(\frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i\right) + \frac{\partial \rho}{\partial t} = 0.$$

Virial theorem

$\overline{T} = -\frac{1}{2}\overline{\sum_i \mathbf{F}_i \cdot \mathbf{r}_i}$. For a system with the interaction potential $V(r) = \alpha r^n$ we have $\overline{T} = \frac{n}{2}\overline{V}$

Definition of the tensor of inertia

$$I = \begin{pmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{pmatrix}$$

Displaced axis theorem

Tensor of inertia about an origin displaced by a constant vector $\mathbf{a}$ is given by
$(I_{\mathbf{a}})_{\alpha\beta} = (I_{\text{c.m.}})_{\alpha\beta} + M(a^2\delta_{\alpha\beta} - a_\alpha a_\beta)$

Moment of inertia about an axis defined by a normal vector

Moment of inertia $I_{\mathbf{n}}$ about an axis that passes through the center of mass and is defined by a normal vector $\mathbf{n}$ is given by (here I is the tensor of inertia of the system):
$I_{\mathbf{n}} = \mathbf{n}^T I \mathbf{n}$

Euler angles and the transformation between the lab and body frame

If $\mathbf{r}'$ is the position in the fixed/lab frame and $\mathbf{r}$ is the position in the body frame then $\mathbf{r} = U_\psi U_\theta U_\phi \mathbf{r}' = U\mathbf{r}'$, where

$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad U_\phi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler equations for a rigid body

$$I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$
$$I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$
$$I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$

Normal frequencies and normal modes of a system of n coupled harmonic oscillators

Generalized eigenvalue problem: $K\mathbf{a}^{(i)} = \omega_i^2 M\mathbf{a}^{(i)}$
Trajectories: $\mathbf{x}(t) = \sum\limits_i \mathbf{a}^{(i)}\mathrm{Re}[e^{i\omega_i t}]$

Unifom continuous string of length L

The equation of motion (wave equation): $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad 0 \le x \le L \quad v = \sqrt{\tau/\rho}$
Normal frequencies: $\omega_n = \frac{n\pi v}{L}$
General solution: $\sum\limits_{n=1}^{\infty} (\beta_n \cos\omega_n t + \gamma_n \sin\omega_n t)\sin\frac{n\pi x}{L}$

Orthogonality of $\sin$ and $\cos$ functions on $(0, L)$ interval

$$\int\limits_0^L \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}\,dx = \frac{L}{2}\delta_{nm} \qquad \int\limits_0^L \cos\frac{n\pi x}{L}\,\cos\frac{m\pi x}{L}\,dx = \frac{L}{2}\delta_{nm} \qquad \int\limits_0^L \sin\frac{n\pi x}{L}\,\cos\frac{m\pi x}{L}\,dx = 0$$

Fourier transform

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

Lorentz transform

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad y' = y \qquad z' = z \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Transformation of velocities in Special Relativity

$$u'_x = \frac{u_x - v}{1 - vu_x/c^2} \qquad u'_y = \frac{u_y}{\gamma(1 - vu_x/c^2)} \qquad u'_z = \frac{u_z}{\gamma(1 - vu_x/c^2)} \quad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Useful integrals

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|\right) + C$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arctan\left[\frac{x}{\sqrt{a^2 - x^2}}\right]\right) + C$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C, \quad a \ne 0$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + C, \quad a \ne 0$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C = \operatorname{arccosh}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C, \quad a \neq 0$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C, \quad a \neq 0$$