① Let us use the method of separation of variables to solve the wave equation

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 q}{\partial t^2}$$

subject to the boundary conditions

$$\left.\frac{\partial q}{\partial x}\right|_{x=0} = \left.\frac{\partial q}{\partial x}\right|_{x=L} = 0$$

and with the initial conditions

$$q(x,0) = f(x) \qquad \dot{q}(x,0) = g(x)$$

So we look for the solution in the form $q(x,t) = X(x)\Theta(t)$

After substituting it into the wave equation we get

$$X''(x)\Theta(t) = \frac{1}{v^2}X(x)\ddot{\Theta}(t) \qquad \text{or} \qquad \frac{X''}{X} = \frac{1}{v^2}\underbrace{\frac{\ddot{\Theta}}{\Theta}}_{-\omega^2} \leftarrow \text{constant}$$

For $\Theta(t)$ we then obtain

$$\ddot{\Theta} + \omega^2\Theta = 0 \qquad \Theta(t) = C\sin\omega t + D\cos\omega t$$

The equation for $X(x)$ is, in turn

$$X'' + \frac{\omega^2}{v^2}X = 0 \qquad X(x) = F\sin\frac{\omega}{v}x + G\cos\frac{\omega}{v}x$$

Here the boundary conditions require that $\left.\frac{\partial X}{\partial x}\right|_{x=0} = \left.\frac{\partial X}{\partial x}\right|_{x=L} = 0$

so $F=0$ while $G\sin\frac{\omega}{v}L = 0 \Rightarrow \omega = \omega_n = \frac{n\pi v}{L}$ $n = 1,2,3\ldots$

With that the general solution of the wave equation can be written as

$$q(x,t) = \sum_n\left(A_n\sin\frac{n\pi vt}{L} + B_n\cos\frac{n\pi vt}{L}\right)\cos\frac{n\pi x}{L}$$

where coefficients A_n and B_n are found using the initial conditions:

$$q(x,0) = f(x) = \sum_n B_n\cos\frac{n\pi x}{L} \qquad B_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\dot{q}(x,0) = g(x) = \sum_n A_n\frac{n\pi v}{L}\cos\frac{n\pi x}{L} \qquad A_n = \frac{2}{n\pi v}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$$

(2) a) Given the definition of the phase and group velocity and the dispersion relation $\omega = \sqrt{gk \tanh(kh)}$

$$v_p \equiv \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh(kh)}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}\tanh(kh)} + \frac{h}{2}\sqrt{gk}\,\frac{1-\tanh^2(kh)}{\sqrt{\tanh(kh)}}$$

So v_p and v_g are quite different. In the open ocean $h \to \infty$ and the expressions simplify to

$$v_p \approx \sqrt{\frac{g}{k}}$$

$$v_g \approx \frac{1}{2}\sqrt{\frac{g}{k}}$$

b) When waves appoach a beach $h \to 0$. In this limit

$$v_p \approx \sqrt{gh}$$

$$v_g \approx \sqrt{gh}$$

so the waves slow down near the beach. While the front of the wave packet slows down, the waves in the back continue coming at a fast pace. This causes the waves to increase in height because the total volume must remain constant (water is incompessible). At some point the height becomes so large that nonlinear effects cause the wave to break.

(3) The speed of light in the medium (as measured in system where the medium is stationary) is $\frac{c}{n}$

Let us introduce a coordinate system, k', that moves with the water. In k' we have $u' = \frac{c}{n}$.

a) The speed u in the laboratory system k is related to u' as follows

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}} \quad \Rightarrow \quad u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

so
$$u = \frac{\frac{c}{n} + v}{1 + \frac{vu'}{c^2}} = \frac{c}{n}\left(\frac{1 + \frac{v}{c}n}{1 + \frac{v}{c}\frac{1}{n}}\right)$$

b) When the medium is stationary $u = \frac{c}{n}$

Thus
$$\Delta u = \frac{c}{n}\left(\frac{1 + \frac{v}{c}n}{1 + \frac{v}{c}\frac{1}{n}} - 1\right) \underset{\frac{v}{c} \ll 1}{\approx} v\left(1 - \frac{1}{n^2}\right)$$

c) In this case $v = c$ and

$$u = \frac{c}{n}\left(\frac{1 + n}{1 + \frac{1}{n}}\right) = c$$