$$J[y] = \int_0^1 \frac{y'^2}{x^3}\,dx \qquad y(0) = 1 \qquad y(1) = 2$$

The integrand is $F(y, y', x) = \frac{y'^2}{x^3}$

The Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$$

$$\frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

$$\frac{2(xy'' - 3y')}{x^4} = 0$$

This yields $xy'' - 3y' = 0$ or $\frac{y''}{y'} = \frac{3}{x}$

Integrating the last expression gives

$$\int \frac{y''}{y'} = 3\int \frac{dx}{x}$$

$$\ln y' = 3\ln x + C$$

$$\ln\left(\frac{y'}{x^3}\right) = C \qquad \text{and} \qquad y' = Cx^3$$

Then we integrate again and get

$$\int dy = \int Cx^3\,dx$$

$$y = Ax^4 + B$$

Constants A and B are determined from the boundary conditions $y(0) = 1$ $y(1) = 2$:

$1 = B$ $\qquad 2 = A + 1 \Rightarrow A = 1$

So the solution is

$$y(x) = x^4 + 1$$