Name: ________________________

PHYS 451 - Quantum Mechanics I
Final Exam (Fall 2014, Instructor: Sergiy Bubin)

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are attached.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. A particle is in the ground state of an infinite 1D square well ($0 < x < a$). Suddenly the well is expanded to a twice larger size in the positive x-direction ($0 < x < 2a$). What is the probability that the energy of the particle is not changed under such an instant expansion?

Problem 2. Find all bound state energies of a particle moving in the field of attractive delta potential,

$$V(x) = -\alpha\delta(x) \qquad (\alpha > 0).$$

Remember that the wave function must be continuous. The first derivative of the wave function, however, may have a discontinuity at the points of singularity. To determine the magnitude of the jump of the first derivative, you may integrate the Schrödinger equation over an infinitely small region around the point of singularity (i.e. from $-\epsilon$ to ϵ).

Problem 3. A particle is placed in the harmonic oscillator potential. Its initial state is a linear combination of the ground and first excited states of the harmonic oscillator:

$$\Psi(x, t = 0) = A[\phi_0(x) + i\phi_1(x)].$$

(a) Find A assuming that states ϕ_0 and ϕ_1 are normalized.

(b) What is the wave function of the particle at time $t > 0$?

(c) Compute the expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{H}\rangle$. Do they change with time?

Problem 4.

(a) Consider a projection operator, $\hat{P}$, whose action on a state, ψ, is defined as

$$\hat{P}\psi = \alpha\phi,$$

where ϕ is some given state and $\alpha = \langle\phi|\psi\rangle$. Is this operator linear? Is it Hermitian? What are its eigenvalues? Compute $\hat{P}^2$ and $\hat{P}^3$.

(b) Consider now another projection operator, $\hat{Q}$, such that $\hat{Q}\psi = \beta\chi$, where $\beta = \langle\chi|\psi\rangle$. Do $\hat{P}$ and $\hat{Q}$ commute? Think well of ϕ and χ.

Problem 5. The Hamiltonian of two particles with spins $s_1 = 3/2$ and $s_2 = 1/2$ is

$$\hat{H} = \lambda(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \hbar\hat{S}_z),$$

where $\hat{S}_z$ is the operator of the z-projection of the total spin ($\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$) and λ is some constant. Find the energies of this system and indicate their degeneracy (if any).

Problem 6. A neutral atom has a single valence electron that is bound in a state with orbital angular momentum quantum number $l = 1$.

(a) What are possible eigenvalues of $\hat{L}^2$ and $\hat{L}_z$?

(b) What is the value of the spin angular momentum quantum number, s? What are the possible eigenvalues of $\hat{S}^2$ and $\hat{S}_z$?

(c) The magnetic moment for the neutral atom is $\boldsymbol{\mu} = -\frac{e}{2m_e}(\mathbf{L} + 2\mathbf{S})$. What are possible eigenvalues of $\hat{\mu}_z$ for this atom?

(d) Suppose a beam of these atom is sent through a Stern-Gerlach apparatus. How many parallel beams will emerge? The Hamiltonian of a particle that has magnetic moment $\boldsymbol{\mu}$ in field $\mathbf{B}$ is $\hat{H} = -\boldsymbol{\mu} \cdot \mathbf{B}$.

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $Y_1^{\mp1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\mp i\phi} = \pm\sqrt{\frac{3}{8\pi}}\,\frac{x \mp iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

Fundamental commutation relations for the components of angular momentum

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

Raising and lowering operators for the z-projection of the angular momentum

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \qquad \text{Action: } \hat{J}_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1 + m_2 = M$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$$

Useful integrals

$$\int_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^k e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0, 1, 2, ...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0, 1, 2, ...)$$

$$\int_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0, 1, 2, ...)$$

Useful trigonometric identities

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$