Name: ____________________

PHYS 451: Quantum Mechanics I, Midterm Exam #1

Instructions:

- All problems are worth the same number of points. The problem for which you get the lowest score will be dropped. Hence, you can pick just 3 problems out of 4, solve them perfectly, and get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information that might be useful is provided on the back side of this sheet.
- No communication with classmates is allowed during the exam.
- Show all your work.
- Staple pages together before submitting and/or number all of them.

Problem 1. The wave function for a particle moving in a potential $V(x)$ is given by

$$\psi(x,t) = \begin{cases} A\,x\exp\left(-Bx - \frac{iCt}{\hbar}\right), & x > 0 \\ 0, & x \leq 0 \end{cases},$$

where A, B, and C are some real constants and $B > 0$.

(a) Is this particle in a state corresponding to a definite energy? If so, what is the energy? If not, why not?

(b) Using what you know about ψ, determine the potential governing this system. Make a qualitative sketch indicating in particular any classically forbidden regions and classical turning points.

Problem 2. A particle is placed in the following potential

$$V(x) = \frac{m\omega^2 x^2}{2} - bx.$$

What is the ground state energy and the wave function of this system?

Problem 3. A particle in the infinite square well of width a has the initial wave function $\Psi(x,0) = A\,\Pi(x)$, where A is a constant and

$$\Pi(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Find:

(a) Normalization constant, A.

(b) $\Psi(x,t)$.

Problem 4. For a quantum system governed by the following Hamiltonian

$$\hat{H} = \frac{p_x^2}{2m} + a|x| \qquad (a > 0)$$

estimate its minimum (i.e. ground state) energy using the uncertainty principle.

The Schrödinger equation

Time-dependent SE: $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$ Stationary SE: $H\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Infinite potential well

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$ where a is the well length and m is the particle mass

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\quad (0 < x < a)$

Matrix elements of the position: $\int\limits_{-\infty}^{+\infty}\phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

Hamiltonian: $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2x^2}{2}$

where m is the particle mass and ω is the angular frequency of the oscillator

Energy levels: $E_n = \hbar\omega(n + \frac{1}{2}),\quad n = 0, 1, 2, ..., \infty$

Two lowest eigenfunctions: $\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},$ where $\alpha = \frac{m\omega}{\hbar}$

Matrix elements of the position: $\int\limits_{-\infty}^{+\infty}\phi_n^*(x)x\,\phi_k(x)dx = \sqrt{\frac{n}{2\alpha}}\,\delta_{n,k+1}$

Heisenberg uncertainty principle

$\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ where standard deviation is defined as $\Delta A \equiv \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

Probability current in 1D

$j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}f(x)e^{-ikx}dx$ $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$

Useful integrals

$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}}\quad(\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$

$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}}\quad(\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$

$\int\limits_0^\infty x^k e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}}\quad(\mathrm{Re}\,\gamma > 0,\ k = 0, 1, 2, ...)$

$\int\limits_{-\infty}^\infty e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}}\quad(\mathrm{Re}\,\beta > 0)$

Delta function

$\int\limits_{-\infty}^\infty f(x)\delta(x - x_0)dx = f(x_0)$ $\delta(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^\infty e^{ikx}dk$

$\delta(-x) = \delta(x)$ $\delta(cx) = \frac{1}{|c|}\delta(x)$