Name: ______________________

PHYS 451: Quantum Mechanics I, Midterm Exam #2

Instructions:

- All problems are worth the same number of points. The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are attached.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning.
- Make sure pages are stapled together before submitting them.

Problem 1. Consider the ground state of the hydrogen atom.

(a) Compute the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, $\langle r \rangle$, and $\langle r^2 \rangle$.

(b) Find the uncertainties Δx and Δr

(c) Compute the probability of finding the electron within a sphere of radius Δr.

Problem 2. An operator $\hat{A}$, representing observable A, has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = (3\phi_1 + 4\phi_2)/5, \qquad \psi_2 = (4\phi_1 - 3\phi_2)/5.$$

(a) Observable A is measured, and the value a_1 is obtained. What is the state of the system immediately after this measurement?

(b) If B is now measured, what are the possible results, and what are their probabilities?

(c) Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be different if you are told the outcome of the B measurement).

Problem 3. The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is given by $\psi(\mathbf{r}) = (x + y + 3z) f(r)$, where $f(r)$ is some function.

(a) Is ψ an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$ operators?

(b) What are the probabilities for the particle to be in various $|l, m\rangle$ states?

Problem 4. An electron is in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the uncertainties ΔS_x, ΔS_y, and ΔS_z.

(d) Confirm that your results for different components of spin are consistent with the uncertainty principle.

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$ Stationary: $H\psi_n = E_n\psi_n$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time $\Delta E\,\Delta t \geq \frac{\hbar}{2}$
General form: $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

The few first radial wave functions for hydrogen ($a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $Y_1^{\mp1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\mp i\phi} = \pm\sqrt{\frac{3}{8\pi}}\,\frac{x\mp iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z$ $[\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x$ $[\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta l_1 and l_2

$$|L\,M\,l_1\,l_2\rangle = \sum_{m_1=-l_1}^{l_1}\sum_{m_2=-l_2}^{l_2}\langle l_1\,m_1\,l_2\,m_2|L\,M\,l_1\,l_2\rangle\,|l_1\,m_1\rangle\,|l_2\,m_2\rangle \qquad m_1+m_2=M$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Useful integrals

$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}}$ $(\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$

$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}}$ $(\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$

$\int\limits_0^\infty x^k e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}}$ $(\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$

$\int\limits_{-\infty}^\infty e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}}$ $(\mathrm{Re}\,\beta > 0)$

$\int\limits_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!}$ $(k = 0,1,2,...)$

$\int\limits_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!}$ $(k = 0,1,2,...)$