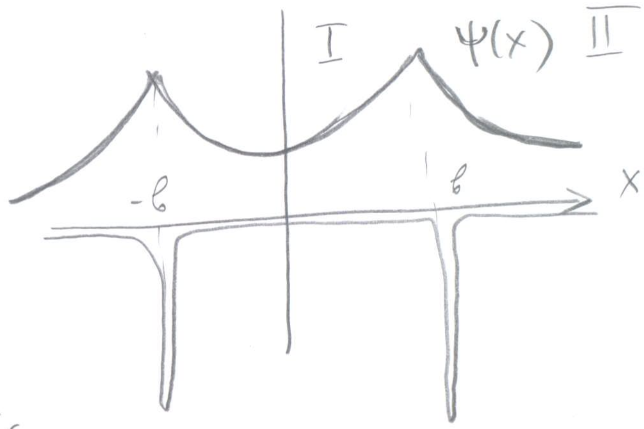
① $V(x)$ is an even function. Therefore the eigenfunction of the Hamiltonian have definite parity. In fact, the ground state has to be even. Since α is positive the bound state energy(ies) are negative, $E<0$.



Since $V=0$ everywhere except $x=\pm b$, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \,, \quad x \neq \pm b$$

In region II the solution is

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x} \,, \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

We have to impose the square integrability of the wave function. That yields $B=0$.

In region I the solution is

$$\psi(x) = Ce^{-\kappa x} + De^{\kappa x} = F\cosh(\kappa x) + G\sinh(\kappa x)$$

Since $\psi(x) = \psi(-x)$ we get $G=0$. Hence,

$$\psi(x) = \begin{cases} Ae^{-\kappa x} \,, & x > b \\ F\cosh(\kappa x) \,, & 0 \le x \le b \end{cases}$$

The continuity of ψ at $x=b$ requires that

$$Ae^{-\kappa b} = F\cosh(\kappa b)$$

Therefore

$$\psi(x) = \begin{cases} Fe^{\kappa b}\cosh(\kappa b)\, e^{-\kappa x} \,, & x > b \\ F\cosh(\kappa x) \,, & 0 \le x \le b \end{cases}$$

Now, constant F can be obtained from the normalization condition:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1 \quad \text{or} \quad \int_0^{\infty} |\psi(x)|^2 dx = \frac{1}{2}$$

With that we get

$$\frac{1}{2} = F^2\left[\int_0^b \cosh^2(\kappa x)\,dx + e^{2\kappa b}\cosh^2(\kappa b)\int_b^\infty e^{-2\kappa x}dx\right] =$$

$$= F^2\left[\frac{1}{4}\int_0^b \left(e^{2\kappa x} + 2 + e^{-2\kappa x}\right)dx + e^{2\kappa b}\cosh^2(\kappa b)\,\frac{e^{-2\kappa b}}{2\kappa}\right] =$$

$$= F^2\left[\frac{1}{4}\left(\frac{e^{2\kappa b}-1}{2\kappa} + 2b + \frac{1-e^{-2\kappa b}}{2\kappa}\right) + \frac{1}{8\kappa}\left(e^{2\kappa b} + 2 + e^{-2\kappa b}\right)\right] =$$

$$= F^2\left[\frac{1}{4\kappa}e^{2\kappa b} + \frac{b}{2} + \frac{1}{4\kappa}\right] \Longrightarrow F = \left(\frac{e^{2\kappa b}+1}{2\kappa} + b\right)^{-1/2}$$

At point $x = b$ the wave function derivative has discontinuity, which follows from integrating the Schrödinger equation over an infinitely small interval $[b-\epsilon, b+\epsilon]$.

$$\psi'(b+\epsilon) - \psi'(b-\epsilon) = -\frac{2m\alpha}{\hbar^2}\psi(b)$$

This yields:

$$-\kappa e^{\kappa b}\cosh(\kappa b)\,e^{-\kappa b} - \kappa\sinh(\kappa b) = -\frac{2m\alpha}{\hbar^2}\cosh(\kappa b)$$

or

$$\kappa\left(1 + \tanh(\kappa b)\right) = \frac{2m\alpha}{\hbar^2}$$

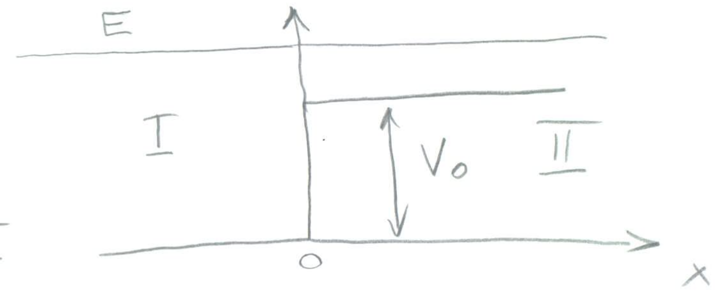
The latter transcendental equation gives κ and E.

In the limit $b \to 0$ it reduces to

$$\kappa = \frac{2m\alpha}{\hbar^2} \quad \text{or} \quad E = -\frac{2m\alpha^2}{\hbar^2}$$

which is a familiar result for a single delta function potential

② Here we consider the Schrödinger equation in two regions – I and II



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

$$\psi_I'' + k\psi_I = 0 \qquad \psi_{II}'' + \varkappa\psi_{II} = 0$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \varkappa = \frac{\sqrt{2m(E-V_0)}}{\hbar}$$

$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad \psi_{II} = Fe^{i\varkappa x} + \underset{\substack{\parallel \\ 0}}{G}\, e^{-i\varkappa x}$$

(no incident wave coming from the right)

The continuity of ψ at $x=0$ gives:

$$A + B = F$$

The continuity of ψ' at $x=0$ gives:

$$ik(A-B) = i\varkappa F$$

Hence

$$A + B = \frac{k}{\varkappa}(A-B) \quad \text{or} \quad A\left(1-\frac{k}{\varkappa}\right) = -B\left(1+\frac{k}{\varkappa}\right)$$

The reflection coefficient is

$$R = \left|\frac{B}{A}\right|^2 = \frac{\left(1-\frac{k}{\varkappa}\right)^2}{\left(1+\frac{k}{\varkappa}\right)^2} = \frac{(k-\varkappa)^2}{(k+\varkappa)^2} = \frac{(k-\varkappa)^4}{(k^2-\varkappa^2)^2} =$$

$$= \frac{\left(\sqrt{E}-\sqrt{E-V_0}\right)^4}{V_0^2}$$

3. According to lecture #5 (or textbook chapter 2.3.1)

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i\hat{p}}{\sqrt{2m\hbar\omega}} \qquad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i\hat{p}}{\sqrt{2m\hbar\omega}}$$

and

$$\hat{H} = \hbar\omega\left(a^+a + \frac{1}{2}\right)$$

$\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^+$ are:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a^+ + a) \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^+ - a)$$

We also know that

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{and} \quad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \text{where } |n\rangle \equiv |\psi_n\rangle$$

With that matrix elements of $\hat{x}$ are:

$$\langle n|\hat{x}|k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|(a^+ + a)|k\rangle$$

Since both a or a^+ when acting on $|n\rangle$ produce an eigenstate of $\hat{H}$ (which are orthogonal to all other eigenstates) we can immediately write

$$\langle n|\hat{x}|k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{k+1}\,\langle n|k+1\rangle + \sqrt{k}\,\langle n|k-1\rangle\right] = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{k+1}\,\delta_{n,k+1} + \sqrt{k}\,\delta_{n,k-1}\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{k+1}\,\delta_{n,k+1} + \sqrt{n+1}\,\delta_{n+1,k}\right]$$

In matrix form this looks as follow (remember that the index begins with 0 for our case):

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Similarly, for $\hat{p}$ we have $\langle n|\hat{p}|k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle n|(a^+ - a)|k\rangle =$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k+1}\,\langle n|k+1\rangle - \sqrt{k}\,\langle n|k-1\rangle\right) = i\sqrt{\frac{m\hbar\omega}{2}}\left[\sqrt{k+1}\,\delta_{n,k+1} - \sqrt{n+1}\,\delta_{n+1,k}\right]$$

$$\hat{p} = \sqrt{\frac{m\hbar\omega}{2}} \begin{pmatrix} 0 & -i\sqrt{1} & 0 & 0 & \cdots \\ i\sqrt{1} & 0 & -i\sqrt{2} & 0 & \cdots \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & \cdots \\ 0 & 0 & i\sqrt{3} & 0 & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

Lastly, for $\hat{H}$ we have:

$$\langle n|\hat{H}|k\rangle = E_k \langle n|k\rangle = E_k \delta_{nk} = \hbar\omega\left(n+\tfrac{1}{2}\right)\delta_{nk}$$

$$\hat{H} = \hbar\omega \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{5}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{7}{2} & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$$

The matrix of any operator is diagonal when computed in the basis of this operator's eigenstates (with eigenvalues on the diagonal)

(4) First, let us compute some auxiliary commutators:

$$[x,H]f = \left[x, \frac{p^2}{2m}\right]f = -\frac{\hbar^2}{2m}\left[x, \frac{d^2}{dx^2}\right]f = -\frac{\hbar^2}{2m}\left(xf'' - \frac{d}{dx}(f+xf')\right) =$$

$$= -\frac{\hbar^2}{2m}\left(xf'' - f' - f' - xf''\right) = -\frac{\hbar^2}{m}f' \qquad \text{So } [x,H] = \frac{i\hbar}{m}p$$

$$[p,H]f = \left[p, \frac{m\omega^2 x^2}{2}\right]f = (-i\hbar)\frac{m\omega^2}{2}\left[\frac{d}{dx}, x^2\right]f = -i\hbar\frac{m\omega^2}{2}\left(\frac{d}{dx}(x^2 f) - x^2 f''\right)$$

$$= -i\hbar\frac{m\omega^2}{2}\left(2xf + x^2 f' - x^2 f''\right) = -i\hbar m\omega^2 x f, \quad \text{So } [p,H] = -i\hbar m\omega^2 x$$

a) $$[X,H] = [x,H]\cos\omega t - [p,H]\frac{\sin\omega t}{m\omega} =$$

$$= \frac{i\hbar}{m}p\cos\omega t + i\hbar\omega x\sin\omega t = \frac{i\hbar}{m}P \neq 0$$

$$[P,H] = [x,H]m\omega\sin\omega t + [p,H]\cos\omega t =$$

$$= i\hbar\omega p\sin\omega t - i\hbar m\omega^2 x\cos\omega t = -i\hbar m\omega^2 X$$

Hence, neither X nor P commute with H

b) For X we have:

$$\frac{d}{dt}\langle X\rangle = \frac{1}{i\hbar}\langle[X,H]\rangle + \left\langle\frac{\partial X}{\partial t}\right\rangle = \frac{1}{i\hbar}\left(\underset{\substack{\shortparallel\\ 0}}{\frac{i\hbar}{m}\langle p\rangle\cos\omega t} + \underset{\substack{\shortparallel\\ 0}}{i\hbar\omega\langle x\rangle\sin\omega t}\right) -$$

$$-\underset{\substack{\shortparallel\\ 0}}{\langle x\rangle\omega\sin\omega t} - \underset{\substack{\shortparallel\\ 0}}{\frac{1}{m}\langle p\rangle\cos\omega t} = 0$$

Similarly, for P we have

$$\frac{d}{dt}\langle P\rangle = \frac{1}{i\hbar}\langle[P,H]\rangle + \left\langle\frac{\partial P}{\partial t}\right\rangle = \frac{1}{i\hbar}\left(\underset{\substack{\shortparallel\\ 0}}{i\hbar\omega\langle p\rangle\sin\omega t} - \underset{\substack{\shortparallel\\ 0}}{i\hbar m\omega^2\langle x\rangle\cos\omega t}\right) +$$

$$+\underset{\substack{\shortparallel\\ 0}}{\langle x\rangle m\omega^2\cos\omega t} - \underset{\substack{\shortparallel\\ 0}}{\langle p\rangle\omega\sin\omega t} = 0$$

Hence, expectation values $\langle X\rangle$ and $\langle P\rangle$ are constant in time, i.e. X and P are conserved quantities

c) We may often hear that conservation requires commutativity with the Hamiltonian. Non-commutativity of X and P with H does not exclude them being conserved, however. This is because X and P contain explicit dependence on t.

(5) a) Both terms in ψ contain spherical harmonics corresponding to $l=1$. Hence

$L^2 = 2\hbar^2$ and P (probability) $= 1$

b) First term has $m_l = 0$, so $L_z = 0$ $\quad P = \left|\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{3}$

Second term has $m_l = 1$, so $L_z = \hbar$ $\quad P = \left|-i\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3}$

c) $S^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 = \frac{3}{4}\hbar^2 \qquad P = 1$

d) First term has $m_s = +\frac{1}{2}$, so $S_z = \frac{\hbar}{2}$ $\quad P = \frac{1}{3}$

Second term has $m_s = -\frac{1}{2}$, so $S_z = -\frac{\hbar}{2}$ $\quad P = \frac{2}{3}$

e) Here we add two angular momenta: $l=1$ and $s=\frac{1}{2}$. Possible outcomes are obviously $j = \frac{3}{2}$ and $\frac{1}{2}$. In order to compute the probabilities of each we need Clebsch-Gordan coefficients for the expasions of $|LM_L\rangle|SM_S\rangle$ in terms of $|JM_J\rangle$. These can be taken from the tables:

$$\begin{array}{ccc} l\ m_l \quad s\ m_s & & j\ m_j \qquad\quad j\ m_j \end{array}$$

$$|1,0\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \tfrac{1}{\sqrt{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

$$|1,1\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$$

Then in the coupled representation

$$\psi = R_{21}(r)\left[\frac{1}{\sqrt{3}}\left(\frac{\sqrt{2}}{\sqrt{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right) - i\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)\right]$$

$$= R_{21}(r)\left[(1-i)\frac{\sqrt{2}}{3}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle - \frac{1+2i}{3}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right]$$

Hence for $j = \frac{3}{2}$ $\quad P = \left|(1-i)\frac{\sqrt{2}}{3}\right|^2 = \frac{4}{9}$ $\quad J^2 = \frac{15}{4}\hbar^2$

for $j = \frac{1}{2}$ $\quad P = \left|\frac{1+2i}{3}\right|^2 = \frac{5}{9}$ $\quad J^2 = \frac{3}{4}\hbar^2$

f) $J_z = \frac{1}{2}\hbar$ (there is only one possibility) $P = 1$

g) $$\rho(\vec{r}) = \psi^\dagger\psi = |R_{21}|^2 \cdot \left(\frac{1}{\sqrt{3}} Y_1^{0*} \quad i\frac{\sqrt{2}}{\sqrt{3}} Y_1^{1*}\right) \begin{pmatrix} \frac{1}{\sqrt{3}} Y_1^0 \\ -i\frac{\sqrt{2}}{\sqrt{3}} Y_1^1 \end{pmatrix} =$$

$$= |R_{21}(r)|^2 \left[\frac{1}{3}|Y_1^0(\theta,\phi)|^2 + \frac{2}{3}|Y_1^1(\theta,\phi)|^2\right] = \frac{1}{3}\frac{1}{24}\frac{1}{a^3}\frac{r^2}{a^2} e^{-\frac{r}{a}} \left[\frac{3}{4\pi}\cos^2\theta + \right.$$

$$\left. + 2\frac{3}{8\pi}\sin^2\theta\right] = \frac{1}{96\pi a^5} r^2 e^{-\frac{r}{a}}$$

h) $$\rho_\uparrow(r) = \int |R_{21}(r)|^2 \frac{1}{3}|Y_1^0(\theta,\phi)|^2 \sin\theta \, d\theta \, d\phi =$$

$$= \frac{1}{3}|R_{21}(r)|^2 = \frac{1}{3}\frac{1}{24a^3}\left(\frac{r}{a}\right)^2 e^{-\frac{r}{a}} = \frac{1}{72 a^5} r^2 e^{-\frac{r}{a}}$$

(6) a) No translational degrees of freedom are involved since electron's speed is parallel to the magnetic field. Thus, the Hamiltonian includes only the interaction of electron's spin with $\vec{B}$:

$$H = -\vec{\mu}\cdot\vec{B} = \frac{e}{m}\vec{B}\cdot\vec{S} = \frac{e}{m}BS_y = \frac{eB}{2m\hbar}\sigma_y = \frac{\hbar\omega}{2}\sigma_y$$

where $\omega = \frac{eB}{m\hbar}$ $\qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The energies are obviously $E_\pm = \pm\frac{\hbar\omega}{2}$ since the eigenvalues of $\sigma_y = \pm 1$

The eigenvectors of σ_y are

$\eta_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$ $\leftarrow$ corresponds to E_+ and the positive projection of spin on the y-axis

$\eta_- = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ $\leftarrow$ corresponds to E_- and the negative projection of spin of the y-axis

Since H is time-independent, the general solution of the time-dependent Schrödinger equation can be written as

$$\chi(t) = a\eta_+ e^{-i\omega t} + b\eta_- e^{+i\omega t} \qquad |a|^2+|b|^2 = 1$$

At time $t=0$ we have

$$\chi(0) = |S_z = -\tfrac{\hbar}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This can be expanded in terms of η_+ and η_-:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\eta_+ + b\eta_- = \frac{1}{\sqrt{2}}\eta_+ + \frac{1}{\sqrt{2}}\eta_-$$

Thus

$$\chi(t) = \frac{1}{\sqrt{2}}\eta_+ e^{-i\omega t} + \frac{1}{\sqrt{2}}\eta_- e^{+i\omega t} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}e^{-i\omega t} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}e^{+i\omega t} =$$

$$= \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$$

b) $\langle S_x \rangle = \chi^{\dagger}(t)\, S_x\, \chi(t) = (-\sin\omega t \quad \cos\omega t)\, \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix} =$

$= \frac{\hbar}{2}\, 2\sin\omega t \cos\omega t = -\frac{\hbar}{2}\sin(2\omega t)$

$\langle S_y \rangle = \chi^{\dagger}(t)\, S_y\, \chi(t) = (-\sin\omega t \quad \cos\omega t)\, \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix} =$

$= 0$

$\langle S_z \rangle = \chi^{\dagger}(t)\, S_z\, \chi(t) = (-\sin\omega t \quad \cos\omega t)\, \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix} =$

$= \frac{\hbar}{2}(\sin^2\omega t - \cos^2\omega t) = -\frac{\hbar}{2}\cos 2\omega t$