PHYS 451: Quantum Mechanics I - Spring 2016
Homework #4, due February 18, in class

Fourier transform, wave packets, Dirac delta-function, transmission and reflection

1. Consider the wave function that has the following form in the coordinate space,

$$\psi(x) = Ae^{-\beta(x-a)^2},$$

where β and a are some positive constants.

   (a) Find the normalization factor A.

   (b) Compute the Fourier transfom of this function, $\tilde{\psi}(k)$. Please do the math by yourself in this part (i.e. do not refer to any tables of integrals). $\tilde{\psi}(k)$ is the wave function in the momentum space. Show that it comes out normalized in the k-space.

   (c) Compute the expectation values $\langle p \rangle$ and $\langle p^2 \rangle$ the usual way, i.e. using the wave function in the coordinate space.

   (d) Keeping in mind the de Broglie relation ($p = \hbar k$) repeat the same procedure in the momentum space, i.e. use $\tilde{\psi}(k)$.

2. Consider the following representation of the Dirac delta function:

$$g(x) = \lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi}\epsilon} \exp\left[-\frac{x^2}{\epsilon^2}\right].$$

Show that this representation satisfies the basic properties of the delta function

   (a) $\int\limits_{-\infty}^{+\infty} g(x)f(x)dx = f(0)$ for any reasonably "nice" function $f(x)$

   (b) $g(x) = g(-x)$

   (c) $xg(x) = 0$

   (d) $g(\alpha x) = \frac{1}{|\alpha|}g(x)$ (here α is a real constant)

   (e) $g'(-x) = -g'(x)$

   (f) $xg'(x) = -g(x)$

   (g) $\tilde{g}(k) = \frac{1}{\sqrt{2\pi}}$ (Fourier transform)

3. At some instant a wave packet has the following momentum distribution (i.e. amplitude function):

$$\phi(k) = \begin{cases} \frac{1}{\sqrt{b}} & , |k| < \frac{b}{2} \\ 0 & , |k| > \frac{b}{2} \end{cases}.$$

   (a) Determine the wave function, $\psi(x)$.

   (b) Verify the uncertainty principle by calculating the product $\Delta p \Delta x$.

4. Find the transmission and reflection coefficients for the potential in the form

$$V(x) = \begin{cases} 0 & , x < 0 \\ V_0 & , x \geq 0 \end{cases},$$

where $V_0 > 0$. Assume that the energy of the particle is greater than V_0.