PHYS 451: Quantum Mechanics I - Spring 2016
Homework #5, due March 10, in class

Creation and annihilation operators, commutators, formalism

1. Using the formalism of the creation/annihilation operators ($a^\dagger$ and a) compute the following general matrix elements in the basis of harmonic oscillator functions $\psi_n(x)$:

   (a) $\langle \psi_n | x | \psi_m \rangle$

   (b) $\langle \psi_n | x^2 | \psi_m \rangle$

   (c) $\langle \psi_n | p | \psi_m \rangle$

   (d) $\langle \psi_n | p^2 | \psi_m \rangle$

   Then find how the uncertainty principle holds for state n, i.e. compute $\Delta x \Delta p$ for state ψ_n.
   Hint: first express x and p in terms of $a^\dagger$ and a, then recall from lecture how $a^\dagger$ and a act on the eigenfunctions of the Hamiltonian.

2. The orbital angular momentum operator is defined as

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix},$$

   where $|...|$ stands for the determinant and $\mathbf{e}_i$ are unit vectors. Find commutators:

   (a) $[\hat{L}_x, \hat{L}_x]$

   (b) $[\hat{L}_x, \hat{L}_y]$

3. Consider a particle that moves in 1D with Hamiltonian $\hat{H} = \frac{p^2}{2m} + V(x)$. Show that the uncertainties of Δp_x and ΔE obey the following inequality:

$$\Delta p_x \Delta E \geq \frac{\hbar}{2} \left| \left\langle \frac{\partial V}{\partial x} \right\rangle \right|.$$

   What does it imply for stationary states?

4. Determine if the following operators are linear, hermitian, nonhermitian, or antihermitian (an operator $\hat{O}$ is called antihermitian if $\hat{O}^\dagger = -\hat{O}$):

   (a) Scaling operator: $\hat{S}_\alpha f(x) = \sqrt{\alpha} f(\alpha x) \quad (\alpha > 0)$

   (b) $\hat{x}\hat{p}_x$

   (c) $i(\hat{A}\hat{B} - \hat{B}\hat{A})$, if it is known that $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$

   (d) $\hat{C} - \hat{C}^\dagger$

   (e) $\alpha\hat{x} - \beta\frac{d}{dx} \quad (\alpha, \beta > 0)$

   (f) Projection operator on some given state $|\alpha\rangle$

5. Prove the following operator relation: $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$