PHYS 451: Quantum Mechanics I - Spring 2016
Homework #7, due Tuesday April 19 in class

Angular momentum, ladder operators addition of angular momenta, spin

1. State $|\psi\rangle$ is an eigenstate of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$, i.e.

$$\hat{\mathbf{L}}^2|\psi\rangle = \hbar^2 l(l+1)|\psi\rangle \quad \text{and} \quad \hat{L}_z|\psi\rangle = \hbar m|\psi\rangle$$

   Find $\langle \hat{L}_x \rangle$ and $\langle \hat{L}_x^2 \rangle$ in this state. *Hint: considering the symmetry with respect to x and y may be helpful.*

2. A spinless particle has the following wave function:

$$\psi = A(x + y + 2z)e^{-\beta r},$$

   where A and β are positive constants and $r = \sqrt{x^2 + y^2 + z^2}$.

   (a) What is the total angular momentum of the particle?

   (b) What is the expectation value of the z-component of the angular momentum?

   (c) What are the probabilities of getting $+2\hbar$ and $+\hbar$ and 0 upon measuring the z-component of the angular momentum?

   (d) What is the probability of finding the particle at angles θ and ϕ (azimuthal and polar angle respectively) in solid angle $d\Omega$?

3. A particle has an orbital angular momentum $l = 2$ and a spin $s = 1$. Its Hamiltonian due to the spin-orbit interaction is $\hat{H} = \frac{\gamma}{\hbar^2}\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$, where γ is a constant. Find the energy levels and degeneracies associated with this spin-orbit interaction.

4. What are the Clebsch-Gordan coefficients involved in the expansion of the following states:

$$|2\,2\,1\,1\rangle, \quad |2\,1\,1\,1\rangle, \quad |2\,0\,1\,1\rangle, \quad |2\,{-1}\,1\,1\rangle, \quad |2\,{-2}\,1\,1\rangle\ ?$$

   Here $|l\,m\,l_1\,l_2\rangle$ stands for a state with a definite value of the total angular momentum (l) and its projection on the z-axis (m) formed by two particles that have orbital angular momenta l_1 and l_2.

   Hint: Start with state $|2\,2\,1\,1\rangle$ or $|2{-}2\,1\,1\rangle$. At some point you might want to use the raising or lowering operator, $\hat{L}_\pm = \hat{L}_{1\pm} + \hat{L}_{2\pm}$, to generate equations containing the unknown coefficients.

5. Regardless of the representation chosen, the four Hermitian matrices, I and σ_i ($i = 1, 2, 3$) satisfy the anti-commutation relation

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\ (i \neq j).$$

   Without resorting to any particular representation (i.e. not using any explicit form) of the matrices, do the following:

   (a) Find $\mathrm{tr}\,(\sigma_i)$.

   (b) Find eigenvalues of σ_i.

   (c) Find $\det(\sigma_i)$.

   (d) Show that the four matrices are linearly independent and an arbitrary 2×2 matrix M can be expressed as their linear combination, i.e. $M = c_0 I + \sum_{i=1}^{3} c_i\sigma_i$. Find the expressions for the coefficients c_i (i=0,1,2,3).