The Cauchy - Schwarz inequality

Consider the usual n-dimensional vector space. In this space we have two vectors v and u. The Cauchy-Schwarz inequality states that

$$|\langle v|u\rangle|^2 \leq \langle v|v\rangle \langle u|u\rangle$$

where $\langle\,|\,\rangle$ stands for an inner (scalar) product. Moreover the two sides of the above equation are equal only when $v = \lambda u$ (i.e. v and u are linearly dependent)
For finite n we can easily show that the above inequality holds true: If we introduce

$$w = u - \frac{\langle u|v\rangle}{\langle v|v\rangle} v$$

then by linearity of the inner product

$$\langle w|v\rangle = \left\langle u - \frac{\langle u|v\rangle}{\langle v|v\rangle} v \,\middle|\, v\right\rangle = \langle u|v\rangle - \frac{\langle u|v\rangle}{\langle v|v\rangle}\langle v|v\rangle = 0$$

vector w is orthogonal to v. Then the norm of

$$u = \frac{\langle u|v\rangle}{\langle v|v\rangle} v + w$$

can be computed easily given this orthogonality of w and v:

$$\|u\|^2 = \left|\frac{\langle u|v\rangle}{\langle v|v\rangle}\right|^2 \|v\|^2 + \|w\|^2 = \frac{|\langle u|v\rangle|^2}{\|v\|^2} + \|w\|^2 \geqslant \frac{|\langle u|v\rangle|^2}{\|v\|^2}$$

Multiplying by $\|v\|^2$ on both sides yields the Cauchy - Schwarz inequality
It turns out the Cauchy-Schwarz inequality also takes place when we deal with a Hilbert space.
If the inner product of two functions is defined as

$$\langle f|g\rangle = \int_a^b f(x)^* g(x)\, dx$$

and if both $f(x)$ and $g(x)$ are square-integrable, then

$$\left|\int_a^b f(x)^* g(x)dx\right| \le \sqrt{\int_a^b |f(x)|^2 dx}\ \sqrt{\int_a^b |g(x)|^2 dx}$$

The uncertainty principle

Back several lectures ago we considered the Heisenberg uncertainty principle for the momentum and position. Let us now prove this very important principle in a more general form.

Suppose we have two observables with corresponding operators $\hat{A}$ and $\hat{B}$.

Recall that

$$(\Delta A)^2 = \langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle = \langle\Psi|(\hat{A}-\langle\hat{A}\rangle)^2\Psi\rangle \xlongequal{\hat{A}\text{ is hermitian}}$$

$$= \langle(\hat{A}-\langle A\rangle)\Psi|(\hat{A}-\langle A\rangle)\Psi\rangle$$

or, if we denote $f \equiv (A-\langle A\rangle)\Psi$,

$$(\Delta A)^2 = \langle f|f\rangle$$

In a similar way

$$(\Delta B)^2 = \langle g|g\rangle \qquad \text{where} \qquad g \equiv (\hat{B}-\langle B\rangle)\Psi$$

Then

$$(\Delta A)^2(\Delta B)^2 = \langle f|f\rangle\langle g|g\rangle \underset{\text{Cauchy-Schwarz}}{\geq} |\langle f|g\rangle|^2$$

For any complex number z the following is true

$$|z|^2 = (\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2 \geq (\mathrm{Im}\,z)^2 = \left[\frac{1}{2i}(z-z^*)\right]^2$$

The general form of the uncertainty principle can be used with other observables, not just x and p. For example, it turns out that different components of angular momenta in quantum mechanics do not commute. Hence they cannot be measured simultaneously.

In general a pair of observables is called incompatible if their operators do not commute.

One important fact about the compatible observables is that they have a shared set of eigenfunctions, while incompatible observables cannot have a complete set of common eigenfunctions.

The energ-time uncertainty principle

It turns out there is an uncertainty relation for the energy and time:

$$\Delta t \, \Delta E \geq \frac{\hbar}{2} \qquad (**)$$

which in its appearace is very similat to the well known

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}$$

It should be noted that in nonrelativistic quantum mechanics time, unlike x, is an independent variable. However, the existence of (**) is not completely counter intuitive. We can recall that

$$\hat{p} \Psi = -i\hbar \frac{\partial}{\partial x} \Psi$$

the Schrödinger equation reads

$$H\Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

from where the analogy becomes more visible.

assuming $\frac{\partial Q}{\partial t} = 0$ we will obtain

$$\Delta H \Delta Q \geq \frac{1}{2} \left| \langle [\hat{H}, \hat{Q}] \rangle \right| = \frac{1}{2} \left| \frac{\hbar}{i} \frac{d\langle Q \rangle}{dt} \right| = \frac{\hbar}{2} \left| \frac{d\langle Q \rangle}{dt} \right|$$

If we define

$$\Delta t \equiv \frac{\Delta Q}{\left| \frac{d\langle Q \rangle}{dt} \right|} \qquad \text{or} \qquad \Delta Q = \left| \frac{d\langle Q \rangle}{dt} \right| \Delta t$$

then we get the energy-time uncertainty principle in the sought form:

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

Δt here represents here the amount of time it takes the expectation value of Q to change by one standard deviation.

<u>Example</u>. If an unstable particle lasts about time Δt before desintegrating then the uncertainty in its energy will be $\Delta E \geq \frac{\hbar}{2\Delta t}$. As we know from special relativity the rest mass is related to the total energy, $E = mc^2$. Thus, if a particle is unstable it is impossible to measure its mass precisely.