Reduction of two-particle into center of mass plus a one-body problem in the case of central potential

We start with a two-particle Hamiltonian:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|\vec{r}_2 - \vec{r}_1|) = -\frac{\hbar^2}{2m_1}\nabla_{\vec{r}_1}^2 - \frac{\hbar^2}{2m_2}\nabla_{\vec{r}_2}^2 + V(|\vec{r}_2 - \vec{r}_1|)$$

If we define the position of the center of mass as

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \quad \text{where} \quad M = m_1 + m_2$$

and the relative position

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

then

$$\nabla_{\vec{r}_1} = \frac{\partial}{\partial\vec{R}}\frac{\partial\vec{R}}{\partial\vec{r}_1} + \frac{\partial}{\partial\vec{r}}\frac{\partial\vec{r}}{\partial\vec{r}_1} = \frac{m_1}{M}\nabla_{\vec{R}} - \nabla_{\vec{r}}$$

$$\nabla_{\vec{r}_2} = \frac{\partial}{\partial\vec{R}}\frac{\partial\vec{R}}{\partial\vec{r}_2} + \frac{\partial}{\partial\vec{r}}\frac{\partial\vec{r}}{\partial\vec{r}_2} = \frac{m_2}{M}\nabla_{\vec{R}} + \nabla_{\vec{r}}$$

The Hamiltonian can be written then as

$$H = -\frac{\hbar^2}{2m_1}\left(\frac{m_1}{M}\nabla_{\vec{R}} - \nabla_{\vec{r}}\right)^2 - \frac{\hbar^2}{2m_2}\left(\frac{m_2}{M}\nabla_{\vec{R}} + \nabla_{\vec{r}}\right)^2 + V(r) =$$

$$= -\frac{\hbar^2}{2M}\nabla_{\vec{R}}^2 - \frac{\hbar^2}{2m_1}\nabla_{\vec{r}}^2 - \frac{\hbar^2}{2m_2}\nabla_{\vec{r}}^2 + V(r) =$$

$$= \underbrace{-\frac{\hbar^2}{2M}\nabla_{\vec{R}}^2}_{H_{CM}} \underbrace{-\frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2 + V(r)}_{H_{relative}}$$

H_{CM} describes free motion of the center of mass while $H_{relative}$ describes the relative motion of particle 2 with respect to particle 1.

Since H_{CM} depends only on $\vec{R}$ and $H_{relative}$ depends only on $\vec{r}$ the eigenfunction of H is a product of the eigenfunctions of H_{CM} and $H_{relative}$:

$$\Psi(\vec{R},\vec{r}) = \psi_{CM}(\vec{R})\,\psi_{relative}(\vec{r})$$

$$H_{CM}\psi_{CM} = E_{CM}\psi_{CM} \qquad H_{relative}\,\psi_{relative} = E_{relative}\,\psi_{relative}$$

and

$$E = E_{CM} + E_{relative}$$

In a similar way we can show that the operator of the angular momentum of a two-particle system can be decomposed into a sum of two operators corresponding to the angular momentum of the center of mass and the angular momentum of the relative motion :

$$\vec{L} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = -i\hbar(\vec{r}_1 \times \nabla_{\vec{r}_1}) - i\hbar(\vec{r}_2 \times \nabla_{\vec{r}_2})$$

$$= -i\hbar\left(\vec{r}_1 \times \left(\frac{m_1}{M}\nabla_{\vec{R}} - \nabla_{\vec{r}}\right)\right) - i\hbar\left(\vec{r}_2 \times \left(\frac{m_2}{M}\nabla_{\vec{R}} + \nabla_{\vec{r}}\right)\right) =$$

$$= -i\hbar\left[\left(\frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2\right) \times \nabla_{\vec{R}} + (\vec{r}_2 - \vec{r}_1) \times \nabla_{\vec{r}}\right] =$$

$$= -i\hbar\left[\vec{R} \times \nabla_{\vec{R}} + \vec{r} \times \nabla_{\vec{r}}\right] = \vec{L}_{CM} + \vec{L}_{relative}$$