StudentID: ______________________

PHYS 451: Quantum Mechanics I – Spring 2016
Instructor: Sergiy Bubin
Midterm Exam #1

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. The wave function of a particle of mass m moving in the field of some 1D potential is given by

$$\psi(x) = A\,\mathrm{sech}(\beta x) = \frac{A}{\cosh(\beta x)},$$

where A and β are positive constants. Find the potential, $V(x)$, assuming that it vanishes at $x = \pm\infty$. Make a qualitative sketch of both the wave function and potential.

Problem 2. A particle is prepared in the state with the following wave function:

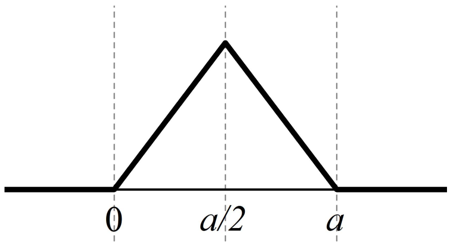
$$\psi(x) = \begin{cases} A\exp(-\alpha x)\exp(\frac{ip_0 x}{\hbar}), & x > 0 \\ 0, & x < 0 \end{cases}$$

where α and p_0 are positive constants.

(a) Calculate the probability that the particle is found in the range $0 \le x \le 1/\alpha$

(b) What is the probability distribution of the momentum in this state? What is the most probable value of the momentum?

Problem 3. A particle of mass m is in the infinite square well ($0 \le x \le a$). Its initial wave function, $\Psi(x, 0)$, is depicted below.



(a) Write $\Psi(x, 0)$ in the analytic form

(b) Find $\Psi(x, t)$

(c) If the energy of the particle is measured at time t, what are the probabilities of getting the values corresponding to the ground and first excited states?

Problem 4. Consider a particle of mass m moving in the potential $V(x) = \gamma x^4$, where γ is a positive constant. Use the uncertainty principle to estimate the ground state energy of this system.

Problem 5. In this problem we will consider a simple model of neutrino oscillations. There are three types of neutrinos in nature: the electron neutrino (ν_e), muon neutrino (ν_μ), and tau neutrino (ν_τ). It has been speculated that each of these neutrinos has a tiny yet finite rest mass, possibly different for each type. Let us suppose that there is a some interaction between these neutrino types, in the absence of which all three types of neutrinos have the same nonzero rest mass M_0. The Hamiltonian of the system (in some units, where the speed of light $c = 1$) can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

where

$$\hat{H}_0 = \begin{pmatrix} M_0 & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & M_0 \end{pmatrix}$$

is the Hamiltonian when there are no interactions between neutrino types and

$$\hat{H}_1 = \begin{pmatrix} 0 & \hbar\omega & \hbar\omega \\ \hbar\omega & 0 & \hbar\omega \\ \hbar\omega & \hbar\omega & 0 \end{pmatrix}$$

is the Hamiltonian describing the interaction. Here we have used the basis states

$$|\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\nu_\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) First assume that there is no interaction, i.e. $\omega = 0$. What happens if at the initial moment of time ($t = 0$) the neutrino was in the state $|\psi(0)\rangle = |\nu_e\rangle$, or $|\psi(0)\rangle = |\nu_\mu\rangle$, or $|\psi(0)\rangle = |\nu_\tau\rangle$?

(b) Now assume that $\omega \neq 0$, i.e. the interaction is present. What is the total energy for each neutrino type?

(c) Now also assume that at the initial time ($t = 0$) the neutrino was in the state $|\psi(0)\rangle = |\nu_e\rangle$. What is the probability as a function of time, that the neutrino will be in each of the other two states?

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Dirac delta function

$\int\limits_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$ $\delta(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{ikx}dk$ $\delta(-x) = \delta(x)$ $\delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx$ $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$

or, in terms of $p = \hbar k$

$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx$ $f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty}\tilde{f}(p)e^{ipx/\hbar}dp$

Useful integrals

$$\int x\sin(\alpha x)\,dx = \frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha}$$

$$\int x^2\sin(\alpha x)\,dx = \frac{2x\sin(\alpha x)}{\alpha^2} - \frac{\left(\alpha^2x^2-2\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^3\sin(\alpha x)\,dx = \frac{3\left(\alpha^2x^2-2\right)\sin(\alpha x)}{\alpha^4} - \frac{x\left(\alpha^2x^2-6\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^4\sin(\alpha x)\,dx = \frac{4x\left(\alpha^2x^2-6\right)\sin(\alpha x)}{\alpha^4} - \frac{\left(\alpha^4x^4-12\alpha^2x^2+24\right)\cos(\alpha x)}{\alpha^5}$$

$$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^{k}e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int\limits_{-\infty}^\infty e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int\limits_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int\limits_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

Useful trigonometric identities

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Useful identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2\, x = 1 \qquad \coth^2 x - \mathrm{csch}^2\, x = 1$$