(1) If $\psi(x) = A\,\mathrm{sech}(\beta x) = \dfrac{A}{\cosh(\beta x)} = \dfrac{2A}{e^{\beta x} + e^{-\beta x}}$ then

$$\frac{d\psi}{dx} = -\frac{A\beta \sinh(\beta x)}{\cosh^2(\beta x)}$$

$$\frac{d^2\psi}{dx^2} = -\frac{A\beta^2}{\cosh(\beta x)} + \frac{2A\beta^2 \sinh^2(\beta x)}{\cosh^3(\beta x)} = \frac{A\beta^2}{\cosh(\beta x)}\left[2\tanh^2(\beta x) - 1\right] = \psi\beta^2\left[2\tanh^2(\beta x) - 1\right]$$

When we substitute ψ'' into the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

we get

$$-\frac{\hbar^2\beta^2}{2m}\left[2\tanh^2(\beta x) - 1\right]\psi + V(x)\psi = E\psi$$

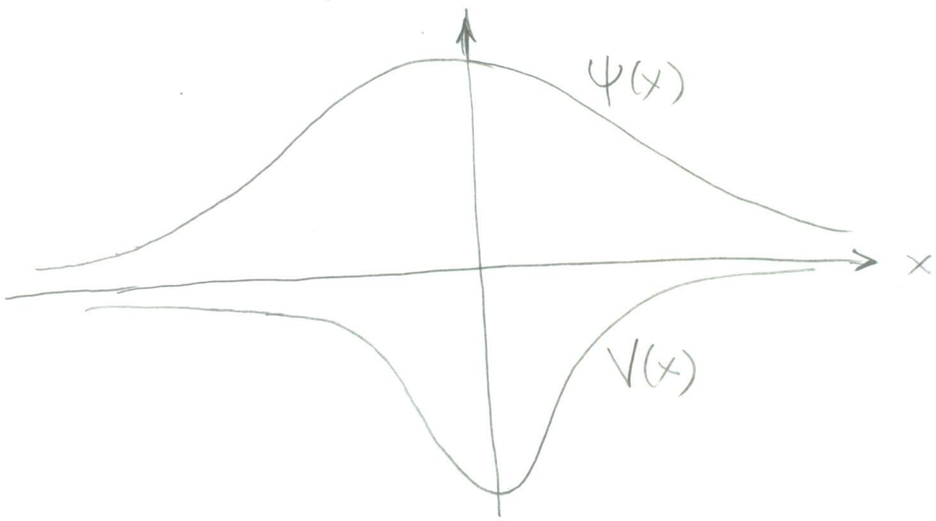
or, after dividing everything by ψ and rearranging terms

$$V(x) = E + \frac{\hbar^2\beta^2}{2m}\left[2\tanh^2(\beta x) - 1\right] = E + \frac{\hbar^2\beta^2}{2m}\left[1 - 2\,\mathrm{sech}^2(\beta x)\right] =$$

$$= E + \frac{\hbar^2\beta^2}{2m} - \frac{\hbar^2\beta^2}{m}\frac{1}{\cosh^2(\beta x)}$$

Since $V(\pm\infty) = 0$ we can get rid of an undetermined additive constant and

$$V(x) = -\frac{\hbar^2\beta^2}{m}\frac{1}{\cosh^2(\beta x)}$$



② First, let us find the normalization constant

$$1=\int_{-\infty}^{+\infty}|\psi(x)|^2dx = |A|^2\int_0^{\infty}e^{-2\alpha x}dx = \frac{|A|^2}{2\alpha} \Rightarrow A=\sqrt{2\alpha}$$

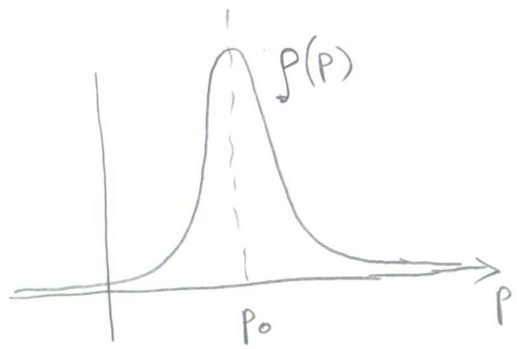
a) $$P(0\le x\le \tfrac{1}{\alpha}) = \int_0^{1/\alpha}|\psi(x)|^2dx = 2\alpha\int_0^{1/\alpha}e^{-2\alpha x}dx = -e^{-2\alpha x}\Big|_0^{1/\alpha} = 1-e^{-2}\approx 0.865$$

b) $\rho(p)=|\tilde{\psi}(p)|^2$ $\quad\tilde{\psi}(p)$ is the Fourier transform

$$\tilde{\psi}(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\psi(x)e^{-\frac{ipx}{\hbar}}dx = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{2\alpha}\int_0^{\infty}e^{-\alpha x}e^{\frac{ip_0x}{\hbar}}e^{-\frac{ipx}{\hbar}}dx =$$

$$=\sqrt{\frac{\alpha}{\pi\hbar}}\int_0^{\infty}e^{-(\alpha+i\frac{p-p_0}{\hbar})x}dx = \sqrt{\frac{\alpha}{\pi\hbar}}\frac{1}{\alpha+i\frac{(p-p_0)}{\hbar}} = \sqrt{\frac{\alpha\hbar}{\pi}}\frac{1}{\hbar\alpha+i(p-p_0)}$$

$$\rho(p)=\tilde{\psi}^*(p)\tilde{\psi}(p)=\frac{\alpha\hbar}{\pi}\frac{1}{\hbar^2\alpha^2+(p-p_0)^2}$$



p_0 is the point where function $\rho(p)$ is maximized. Thus, p_0 is the most probable value of the momentum

(3) a) First, let us determine the analytic form of the wave function:

$$\Psi(x,0) = \begin{cases} \alpha x \\ -\beta x + \gamma \end{cases}$$ where $\alpha, \beta,$ and γ are positive constants

Given the symmetry of $\Psi(x,0)$ with respect to the point $x = \frac{a}{2}$ we can immediately say

$$1 = 2\int_0^{a/2} (\alpha x)^2 \, dx = 2\alpha^2 \frac{x^3}{3}\Big|_0^{a/2} = \frac{2}{3}\alpha^2\left(\frac{a}{2}\right)^3 = \frac{\alpha^2 a^3}{12}$$

so $\alpha = \frac{2\sqrt{3}}{a^{3/2}}$

It is easy to see that $\beta = \alpha$ and $\gamma = \alpha a$. However, we will actually not need β and γ later in this problem if we utilize the symmetry.

b) Since $V(x)$ is time-independent we can expand $\Psi(x,t)$ as

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \phi_n(x) e^{-\frac{iE_n t}{\hbar}}$$

where $\phi_n(x)$ and E_n are the eigenfunctions and energy eigenvalues of the particle in the infinite square well

$$c_n = \int_0^a \phi_n^*(x) \Psi(x,0) \, dx \qquad n = 1, 2, 3, \ldots, \infty$$

Now, $\Psi(x,0)$ is symmetric with respect to the point $x = \frac{a}{2}$, while $\phi_n(x)$ are either symmetric (n is odd) or antisymmetric (n is even) with respect to the same point. Hence, taking into account the symmetry of the integrand we get:

n is even: $\quad c_n = 0$

n is odd: $\quad c_n = 2\int_0^{a/2} \sqrt{\frac{2}{a}} \sin\frac{\pi n x}{a}\, \Psi(x,0)\, dx =$

$$= \frac{2^{3/2}}{a^{1/2}} \int_0^{a/2} \sin\frac{\pi n x}{a}\, \frac{2\sqrt{3}}{a^{3/2}} x\, dx = \frac{2^{5/2}\sqrt{3}}{a^2} \int_0^{a/2} x \sin\frac{\pi n x}{a}\, dx =$$

$$= \frac{2^{5/2}\sqrt{3}}{a^2} \left[\frac{\sin\left(\frac{\pi n x}{a}\right)}{\left(\frac{\pi n}{a}\right)^2} - \frac{x\cos\left(\frac{\pi n x}{a}\right)}{\frac{\pi n}{a}} \right] \Bigg|_0^{a/2} = 2^{5/2}\sqrt{3} \left[\frac{\sin\frac{\pi n}{2}}{\pi^2 n^2} - \frac{\cos\frac{\pi n}{2}}{2\pi n} \right]$$

or

$$c_n = \begin{cases} 2^{5/2}\sqrt{3}\, \frac{(-1)^{\frac{n-1}{2}}}{\pi^2 n^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

c)

$$P(E=E_1) = |c_1|^2 = \frac{96}{\pi^4}$$

$$P(E=E_2) = |c_2|^2 = 0$$

(4) The Hamiltonian of our system is $H = \frac{p^2}{2m} + \gamma x^4$

Let us consider the expectation value of H in its ground state:

$$\langle H \rangle = \frac{1}{2m}\langle p^2 \rangle + \gamma \langle x^4 \rangle$$

Recall that $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$

For a stationary state $\langle p \rangle = 0$, so $\langle p^2 \rangle = (\Delta p)^2$

Now for any quantity A the following inequality holds

$$\langle A^2 \rangle \geq \langle A \rangle^2$$

So if we put $A \equiv x^2$ we get

$$\langle x^4 \rangle \geq \langle x^2 \rangle^2$$

Also

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

from which it follows that $\langle x^2 \rangle \geq (\Delta x)^2$. So

$$\langle x^4 \rangle \geq (\Delta x)^4$$

Going back to $\langle H \rangle$ we obtain

$$\langle H \rangle \geq \frac{(\Delta p)^2}{2m} + \gamma (\Delta x)^4$$

The uncertainty principle says $\Delta p \Delta x \geq \frac{\hbar}{2}$ or $\Delta p \geq \frac{\hbar}{2\Delta x}$

With that we have

$$\langle H \rangle \geq \frac{\hbar^2}{8m(\Delta x)^2} + \gamma (\Delta x)^4$$

Now we just need to find the minimum of the last expression:

$$\frac{\partial \langle H \rangle}{\partial (\Delta x)} = 0 \Rightarrow -\frac{\hbar^2}{4m(\Delta x)^3} + 4\gamma(\Delta x)^3 = 0 \Rightarrow \Delta x = \left(\frac{\hbar^2}{16 m\gamma}\right)^{1/6}$$

and at that Δx the expectation value of H is

$$\langle H \rangle = \frac{\hbar^2}{8m}\left(\frac{16m\gamma}{\hbar^2}\right)^{1/3} + \gamma\left(\frac{\hbar^2}{16m\gamma}\right)^{2/3} = \frac{3}{16^{2/3}}\left(\frac{\hbar^2\sqrt{\gamma}}{m}\right)^{2/3}$$

(5) a) If there is no interaction then $|\nu_e\rangle$, $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$ are the eigenstates of the Hamiltonian $H = H_0$ corresponding to a triple degenerate eigenvalue M_0. If the neutrino starts in one of the basis states then it stays in it, i.e. there are no neutrino oscillations. For example, if

$$|\psi(0)\rangle = 1\cdot|\nu_e\rangle + 0\cdot|\nu_\mu\rangle + 0|\nu_\tau\rangle$$

then

$$|\psi(t)\rangle = \underbrace{1}_{C_e(t)}|\nu_e\rangle e^{-\frac{iM_0 t}{\hbar}} + \underbrace{0}_{C_\mu(t)}|\nu_\mu\rangle e^{-\frac{iM_0 t}{\hbar}} + \underbrace{0}_{C_\tau(t)}|\nu_\tau\rangle e^{-\frac{iM_0 t}{\hbar}}$$

$$P_{e\to\mu} = |C_\mu(t)|^2 = 0 \qquad P_{e\to\tau} = |C_\tau(t)|^2 = 0$$

b) If there is interaction, the total Hamiltonian is no longer diagonal in the basis $|\nu_e\rangle$, $|\nu_\mu\rangle$, and $|\nu_\tau\rangle$. Let us find its eigenvalues and eigenstates:

$$\det\begin{pmatrix} M_0-E & \hbar\omega & \hbar\omega \\ \hbar\omega & M_0-E & \hbar\omega \\ \hbar\omega & \hbar\omega & M_0-E \end{pmatrix} = 0 \Rightarrow (M_0-E)^3 + 2(\hbar\omega)^3 - 3(M_0-E)(\hbar\omega)^2 = 0$$

if we denote $\varepsilon \equiv M_0 - E$ and $\gamma \equiv \hbar\omega$ the characteristic equation becomes $\varepsilon^3 - 3\varepsilon\gamma^2 + 2\gamma^3 = 0$

The solutions of it are

$$\varepsilon_1 = \gamma \qquad \varepsilon_2 = \gamma \qquad \varepsilon_3 = -2\gamma$$

So

$$E_1 = M_0 - \hbar\omega \qquad E_2 = M_0 - \hbar\omega \qquad E_3 = M_0 + 2\hbar\omega$$

for $E_{1,2} = M - \hbar\omega$ we have

$$\begin{pmatrix} \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma \\ \gamma & \gamma & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$x + y + z = 0$

if $x = 1$ $\quad z = -y - 1$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ These two eigenvectors (corresponding to a degenerate eigenvalue) are not orthogonal. But we can make up orthogonal linear combinations.

the orthogonal eigenvectors are

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad |u_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $E_3 = M + 2\hbar\omega$ we have

$$\begin{pmatrix} -2\gamma & \gamma & \gamma \\ \gamma & -2\gamma & \gamma \\ \gamma & \gamma & -2\gamma \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$-2x + y + z = 0$

$x - 2y + z = 0$

if $x = 1$ $\quad y + z = 2 \quad y = 1$

$2y - z = 1 \quad z = 1$

so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the normalized eigenvector is $|u_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

c)

$$|\psi(t)\rangle = c_1|u_1\rangle e^{-\frac{iE_1 t}{\hbar}} + c_2|u_2\rangle e^{-\frac{iE_2 t}{\hbar}} + c_3|u_3\rangle e^{-\frac{iE_3 t}{\hbar}}$$

$|c_1|^2 + |c_2|^2 + |c_3|^2 = 1 \leftarrow$ normalization condition

Now we can express $|\nu_e\rangle$ in terms of $|u_i\rangle$:

$$|\nu_e\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{6}}|u_2\rangle + \frac{1}{\sqrt{3}}|u_3\rangle$$

So if $|\psi(0)\rangle = |\nu_e\rangle$ then

$$c_1 = \frac{1}{\sqrt{2}} \qquad c_2 = \frac{1}{\sqrt{6}} \qquad c_3 = \frac{1}{\sqrt{3}}$$

Then

$$P_{e\to\mu}(t) = |\langle \nu_\mu | \psi(t)\rangle|^2$$

$$P_{e\to\tau}(t) = |\langle \nu_\tau | \psi(t)\rangle|^2$$

$$\psi(t) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} e^{-i\frac{E_1}{\hbar}t} + \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{pmatrix} e^{-i\frac{E_1}{\hbar}t} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} e^{-i\frac{E_3}{\hbar}t}$$

where $\frac{E_1}{\hbar} = \frac{M_0}{\hbar} - \omega \qquad \frac{E_3}{\hbar} = \frac{M_0}{\hbar} + 2\omega$.

$$P_{e\to\mu}(t) = \left| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3}\left(e^{-\frac{iE_1 t}{\hbar}} + e^{-\frac{iE_3 t}{\hbar}}\right) \\ \frac{1}{3}\left(-e^{-\frac{iE_1 t}{\hbar}} + e^{-\frac{iE_3 t}{\hbar}}\right) \\ \frac{1}{3}\left(-e^{-\frac{iE_1 t}{\hbar}} + e^{-\frac{iE_3 t}{\hbar}}\right) \end{pmatrix} \right|^2 =$$

$$= \frac{1}{9}\left| e^{-\frac{iE_3 t}{\hbar}} - e^{-\frac{iE_1 t}{\hbar}} \right|^2 = \frac{1}{9}\left| e^{-2i\omega t} - e^{i\omega t}\right|^2 =$$

$$= \frac{1}{9}\left|1 - e^{3i\omega t}\right|^2 = \frac{2}{9}\left(1 - \cos 3\omega t\right) \leftarrow$$ oscillating probability

Similarly

$$P_{e\to\tau}(t) = \frac{2}{9}\left(1 - \cos 3\omega t\right)$$

Finally

$$P_{e\to e}(t) = \frac{5}{9} + \frac{4}{9}\cos 3\omega t$$