StudentID: ______________________

PHYS 451: Quantum Mechanics I – Spring 2016
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Midterm Exam #2

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1.

(a) Consider the ground state of a hydrogen-like atom with $Z = 2$. If the distance between the electron and nucleus were measured, what would be the most likely outcome of this measurement?

(b) Suddenly something happens to the nucleus and Z becomes 1 (the change occurs at the time scale that is much faster than the time scale of the electronic motion in the atom). What is the probability that the system remains in the ground state?

Problem 2.

(a) By considering the rate of change of operator xp_x derive the virial theorem in 1D:

$$2\langle T\rangle = \left\langle x\frac{\partial V}{\partial x}\right\rangle,$$

where T and V are the kinetic and potential energy respectively.

(b) Generalize the theorem to the 3D case, i.e. consider operator $\mathbf{r}\cdot\mathbf{p}$

(c) A particle of mass m moves in a central potential $V(r) = \alpha \ln \frac{r}{r_0}$, where α and r_0 are some positive constants. Using the result obtained in part (b) show that for any eigenstate of the system with such a potential the mean square "velocity" ($\mathbf{v} = \mathbf{p}/m$) is the same and find the expression for it.

Problem 3.

(a) Consider a particle with spin 1. Find the matrix representation of operators S_x, S_y, and S_z.

(b) If the Hamiltonian of a particle with spin 1 is (A and B are some constants)

$$H = AS_z + BS_x^2,$$

what are the energies of this system?

Problem 4. Consider an electron in a uniform magnetic field $\mathbf{B} = (0, 0, B)$. A measurement at time $t = 0$ gave the positive projection of the spin on the z-axis. What is the state vector for the spin at $t > 0$? What is the average spin polarization (i.e. expectation value) along the x-direction at $t > 0$?

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well $(0 \leq x \leq a)$

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions R_{nl} for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $\qquad$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $\qquad$ $Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\,\frac{x\pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\qquad$ $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ $\qquad$ $[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$ $\qquad$ $[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ $\qquad$ Action: $\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1+m_2=M$$

Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\qquad$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Electron in a magnetic field

Hamiltonian: $H = \frac{e}{m}\mathbf{B}\cdot\mathbf{S} = \mu_\mathrm{B}\,\mathbf{B}\cdot\boldsymbol{\sigma}$ $\qquad$ $\mu_\mathrm{B} = \frac{e\hbar}{2m}$

Dirac delta function

$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$ $\qquad$ $\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}dk$ $\qquad$ $\delta(-x) = \delta(x)$ $\qquad$ $\delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx$ $\qquad$ $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$

or, in terms of $p = \hbar k$

$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx$ $\qquad$ $f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\tilde{f}(p)e^{ipx/\hbar}dp$

Useful integrals

$$\int x\sin(\alpha x)\,dx = \frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha}$$

$$\int x^2\sin(\alpha x)\,dx = \frac{2x\sin(\alpha x)}{\alpha^2} - \frac{\left(\alpha^2x^2-2\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^3\sin(\alpha x)\,dx = \frac{3\left(\alpha^2x^2-2\right)\sin(\alpha x)}{\alpha^4} - \frac{x\left(\alpha^2x^2-6\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^4\sin(\alpha x)\,dx = \frac{4x\left(\alpha^2x^2-6\right)\sin(\alpha x)}{\alpha^4} - \frac{\left(\alpha^4x^4-12\alpha^2x^2+24\right)\cos(\alpha x)}{\alpha^5}$$

$$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta>0,\ k=0,1,2,...)$$

$$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta>0,\ k=0,1,2,...)$$

$$\int\limits_0^\infty x^{k}e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma>0,\ k=0,1,2,...)$$

$$\int\limits_{-\infty}^\infty e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta>0)$$

$$\int\limits_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k=0,1,2,...)$$

$$\int\limits_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k=0,1,2,...)$$

Useful trigonometric identities

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

Useful identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2\, x = 1 \qquad \coth^2 x - \mathrm{csch}^2\, x = 1$