a) The wave function must be normalized at time $t=0$ (as well as at any other instant), so

$$1 = \int |\Psi(x,0)|^2 dx = |A|^2 \int_0^a [2\phi_1^* - i\phi_2^*][2\phi_1 - i\phi_2]\, dx =$$

$$= |A|^2 \Big(4 \underbrace{\int |\phi_1|^2 dx}_{1} + \underbrace{\int |\phi_2|^2 dx}_{1} + 2i \underbrace{\int \phi_2^* \phi_1 \, dx}_{0} - 2i \underbrace{\int \phi_0^* \phi_2 \, dx}_{0} \Big) =$$

$$= 5|A|^2 \implies A = \frac{1}{\sqrt{5}} \quad \text{(up to an arbitrary phase factor)}$$

b) Given the fact that the potential is time independent we can write

$$\Psi(x,t) = \frac{2}{\sqrt{5}} \phi_1(x) e^{-\frac{iE_1 t}{\hbar}} - \frac{i}{\sqrt{5}} \phi_2(x) e^{-\frac{iE_2 t}{\hbar}}$$

where $E_1 = \frac{\pi^2 \hbar^2}{2ma^2} \equiv \hbar\omega$ and $E_2 = \frac{4\pi^2\hbar^2}{2ma^2} = 4E_1 = 4\hbar\omega$

The probability density is then

$$|\Psi(x,t)|^2 = \left(\frac{2}{\sqrt{5}} \phi_1^* e^{i\omega t} + \frac{i}{\sqrt{5}} \phi_2^* e^{4i\omega t} \right) \left(\frac{2}{\sqrt{5}} \phi_1 e^{-i\omega t} - \frac{i}{\sqrt{5}} \phi_2 e^{-4i\omega t} \right) =$$

$$= \frac{1}{5}\left(4|\phi_1|^2 + |\phi_2|^2 - 2i\phi_1^* \phi_2 e^{-3i\omega t} + 2i \phi_2^* \phi_1 e^{3i\omega t} \right) =$$

$$= \frac{1}{5}\left(4\phi_1^2 + \phi_2^2 - 4\phi_1\phi_2 \frac{e^{-3i\omega t} - e^{-3i\omega t}}{2i} \right) = \underbrace{\frac{1}{5}\left(4\phi_1^2 + \phi_2^2 - 4\phi_1\phi_2 \sin[3\omega t] \right)}_{\text{real positive function}}$$

c) $\Psi(x,t)$ is not a stationary state because the probability density (and, consequently, most expectation values) changes with time

d)

$$\langle H \rangle = \int \left(\frac{2}{\sqrt{5}} \phi_1 e^{i\omega t} + \frac{i}{\sqrt{5}} \phi_2 e^{4i\omega t} \right) H \left(\frac{2}{\sqrt{5}} \phi_1 e^{-i\omega t} - \frac{i}{\sqrt{5}} \phi_2 e^{-4i\omega t} \right) dx =$$

$$= \int \left(\frac{2}{\sqrt{5}} \phi_1 e^{i\omega t} + \frac{i}{\sqrt{5}} \phi_2 e^{4i\omega t} \right) \left(E_1 \frac{2}{\sqrt{5}} \phi_1 e^{-i\omega t} - E_2 \frac{i}{\sqrt{5}} \phi_2 e^{-4i\omega t} \right) dx$$

The cross terms in the last integral are zeros due to the orthogonality of the eigenfunctions. In the end we obtain

$$\langle H \rangle = \frac{4}{5}E_1 + \frac{1}{5}E_2 = \frac{8}{5}E_1 = \frac{8}{5}\hbar\omega$$

Notice that $\langle H \rangle$ is time independent

e) $\langle x \rangle = \int \left(\frac{2}{\sqrt{5}}\phi_1 e^{i\omega t} + \frac{i}{\sqrt{5}}\phi_2 e^{4i\omega t}\right) x \left(\frac{2}{\sqrt{5}}\phi_1 e^{-i\omega t} - \frac{i}{\sqrt{5}}\phi_2 e^{-4i\omega t}\right) dx$

Here the cross terms do not vanish (this is generally the case unless functions ϕ_i have some special symmetry) With that we have (see the formulae in the appendix)

$$\langle x \rangle = \frac{1}{5}\left(4\underbrace{\int \phi_1 x \phi_1\, dx}_{a/2} + \underbrace{\int \phi_2 x \phi_1\, dx}_{a/2} - 2ie^{-3i\omega t}\underbrace{\int \phi_1 x \phi_2\, dx}_{-\frac{16a}{9\pi^2}} + 2ie^{3i\omega t}\underbrace{\int \phi_2 x \phi_1\, dx}_{-\frac{16a}{9\pi^2}}\right) =$$

$$= \frac{a}{2} + \frac{64a}{45\pi^2}\,\frac{e^{3i\omega t} - e^{-3i\omega t}}{2i} = \frac{a}{2}\left(1 + \frac{128}{45\pi^2}\sin 3\omega t\right)$$

Sanity check: $\frac{128}{45\pi^2} \approx 0.288 < 1$ so that $\langle x \rangle$ oscillates well within 0 and a