StudentID: ______________________

PHYS 451: Quantum Mechanics I – Spring 2017
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Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Find the eigenvalues λ and eigenfunctions ψ_λ of the operator $\hat{A} = \alpha\hat{p}_x + \beta\hat{x}$, where α and β are real parameters. Is the spectrum of the operator discrete or continuous? Are any of the eigenvalues degenerate? Show that the eigenfunctions are mutually orthogonal and make sure to normalize them properly.

Problem 2. A particle of mass m moves in two dimensions and is subject to harmonic forces, which can be put in the form of the following Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}(x^2 + y^2) + \beta m\omega^2 xy,$$

where β is a real number.

(a) What condition must β satisfy so that the particle has discrete energy levels?

(b) Given that the above condition is satisfied, find the energy levels. Are any of them degenerate? If so, what is the degeneracy?

Problem 3. Consider a quantum harmonic oscillator in 1D. Algebraically, using the creation and annihilation operators (but without using any explicit form of the wave functions), construct a linear combination of states $|0\rangle$ and $|1\rangle$ such that the expectation value $\langle x\rangle$ is as large as possible. What is this expectation value?

Problem 4. A system with spin 1/2 is known to be in the eigenstate corresponding to the positive projection of the spin on axis $\mathbf{n}$, where $\mathbf{n}$ is a unit vector that lies in the xz plane and makes angle γ with the z axis.

(a) If S_x is measured, what is the probability of getting the value $+\hbar/2$?

(b) Evaluate the uncertainty of S_x.

Problem 5. Consider a particle with spin one ($s = 1$). The particle is placed in a uniform external magnetic field $\mathbf{B} = (B, 0, 0)$. Assume that the gyromagnetic ratio that relates the particle spin to its magnetic moment is g, i.e. $\boldsymbol{\mu} = g\mathbf{S}$.

(a) Find the explicit form of spin matrices in the basis $|sm\rangle$ of eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$.

(b) If the particle is initially ($t = 0$) in state $|1\,1\rangle$, find the evolved state at $t > 0$.

(c) What is the probability of finding the particle in state $|1\,{-1}\rangle$ at $t > 0$?

Problem 6. Let $x(t)$ be coordinate operator for a free particle in 1D in the Heisenberg picture. Evaluate the commutator $[x(t), x(0)]$.

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1,2,...,\infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{n}\,\delta_{k,n-1}\right)$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions R_{nl} for the hydrogen atom ($a=\frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10}=2a^{-3/2}\,e^{-\frac{r}{a}}$ $\quad$ $R_{20}=\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $\quad$ $R_{21}=\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0=\frac{1}{\sqrt{4\pi}}$ $\quad$ $Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta=\sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $\quad$ $Y_1^{\pm1}=\mp\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\pm i\phi}=\mp\sqrt{\frac{3}{8\pi}}\,\frac{x\pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\quad$ $\hat{L}_z=-i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x,\hat{J}_y]=i\hbar\hat{J}_z$ $\quad$ $[\hat{J}_y,\hat{J}_z]=i\hbar\hat{J}_x$ $\quad$ $[\hat{J}_z,\hat{J}_x]=i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm=\hat{J}_x\pm i\hat{J}_y$ $\quad$ Action: $\hat{J}_\pm|j,m\rangle=\hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$|J\,M\,j_1\,j_2\rangle=\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle$ $\quad$ $m_1+m_2=M$

$|j_1\,m_1\rangle\,|j_2\,m_2\rangle=\sum_{J=|j_1-j_2|}^{j_1+j_2}\langle JMj_1j_2|j_1m_1j_2m_2\rangle\,|J\,M\,j_1\,j_2\rangle$ $\quad$ $M=m_1+m_2$

Pauli matrices

$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad$ $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\quad$ $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Electron in a magnetic field

Hamiltonian: $H=-\boldsymbol{\mu}\cdot\mathbf{B}=-\gamma\,\mathbf{B}\cdot\mathbf{S}=\frac{e}{m}\mathbf{B}\cdot\mathbf{S}=\mu_\mathrm{B}\,\mathbf{B}\cdot\boldsymbol{\sigma}$
here $e>0$ is the magnitude of the electron electric charge and $\mu_\mathrm{B}=\frac{e\hbar}{2m}$

Bloch theorem for periodic potentials $V(x+a)=V(x)$

$\psi(x)=e^{ikx}u(x)$, where $u(x+a)=u(x)$ $\quad$ Equivalent form: $\psi(x+a)=e^{ika}\psi(x)$

Density matrix $\hat{\rho}$

$\hat{\rho}=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, $\quad$ where $\sum_i p_i=1$

Expectation value of some observable A: $\langle\hat{A}\rangle=\sum_i p_i\langle\psi_i|\hat{A}|\psi_i\rangle=\mathrm{tr}(\hat{\rho}\hat{A})$, where $\mathrm{tr}(\hat{\rho})=1$

Time evolution operator

$$\hat{U}(t_f,t_i)=\hat{\mathcal{T}}\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f}\hat{H}(t)dt\right]=1+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_1}dt_2\ldots\int_{t_i}^{t_{n-1}}dt_n\hat{H}(t_1)\hat{H}(t_2)\ldots\hat{H}(t_n)$$

In particular, $\hat{U}(t_f,t_i)=\exp\left[-\frac{i}{\hbar}\hat{H}(t_f-t_i)\right]$ when $\hat{H}\neq\hat{H}(t)$

Schrödinger, Heisenberg and interaction pictures

$\psi_H=\hat{U}^{-1}\psi_S$, $\psi_H=\psi_S(t{=}0)$, $\hat{A}_H=\hat{U}^{-1}\hat{A}_S\hat{U}$, $i\hbar\frac{\hat{A}_H}{dt}=[\hat{A}_H,\hat{H}]+i\hbar\frac{\partial\hat{A}_H}{\partial t}$, $\frac{\partial\hat{A}_H}{\partial t}\equiv\hat{U}^{-1}\frac{\partial\hat{A}_S}{\partial t}\hat{U}$
If $\hat{H}=\hat{H}_0+\hat{V}(t)$, then
$\psi_I=\hat{U}_0^{-1}\psi_S$, $\hat{U}_0=\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right]$, $\hat{A}_I=\hat{U}_0^{-1}\hat{A}_S\hat{U}_0$, $i\hbar\frac{\partial\psi_I}{\partial t}=\hat{V}_I\psi_I$

$\psi_I(t)=\psi_I(0)+\frac{1}{i\hbar}\int_0^t\hat{V}_I(t')\psi_I(t')dt'$

Dirac delta function

$$\int\limits_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \tfrac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \tfrac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \tfrac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \tfrac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} \tilde{f}(p)e^{ipx/\hbar}dp$$

Useful integrals

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\arctan\left[\tfrac{x}{\sqrt{a^2-x^2}}\right]\right)$$

$$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\tfrac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \tfrac{1}{2}\tfrac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int\limits_0^\infty x^k e^{-\gamma x}dx = \tfrac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int\limits_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\tfrac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int\limits_0^\pi \sin^{2k}x\,dx = \pi\tfrac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int\limits_0^\pi \sin^{2k+1}x\,dx = \tfrac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

$$\int\limits_0^{2\pi} \cos m\phi\, e^{in\phi}\,dx = \pi(\delta_{m,n} + \delta_{m,-n}) \quad (m,n = 0,\pm1,\pm2,...)$$

Useful trigonometric identities

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Useful identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2 x = 1 \qquad \coth^2 x - \mathrm{csch}^2 x = 1$$