

① The operator is  $A = \alpha p_x + \beta x = -i\hbar \alpha \frac{d}{dx} + \beta x$

We need to solve the following eigenvalue problem:

$$-i\hbar \alpha \frac{d\psi_\lambda}{dx} + \beta x \psi_\lambda = \lambda \psi_\lambda \quad \text{or} \quad \frac{d\psi_\lambda}{dx} = \frac{i}{\hbar \alpha} (\lambda - \beta x) \psi_\lambda$$

The integration of the latter equation yields

$$\psi_\lambda = C_\lambda \exp \left[ \frac{i}{\hbar \alpha} \left( \lambda x - \frac{\beta x^2}{2} \right) \right] \quad \text{where } C_\lambda \text{ is an integration constant that can be found from normalization}$$

The solutions of the eigenvalue problem  $A\psi_\lambda = \lambda \psi_\lambda$  exist for any  $\lambda$ . Therefore, the spectrum is continuous. There is no degeneracy — each  $\lambda$  corresponds to a single  $\psi_\lambda$  only.

The normalization condition in the case of a continuous spectrum is

$$\begin{aligned} \delta(\lambda - \lambda') &= \langle \psi_\lambda | \psi_{\lambda'} \rangle = \int_{-\infty}^{+\infty} C_\lambda^* C_{\lambda'} \exp \left[ \frac{i}{\hbar \alpha} (\lambda' - \lambda)x \right] dx \\ &= 2\pi C_\lambda^* C_{\lambda'} \delta \left( \frac{\lambda' - \lambda}{\hbar \alpha} \right) = 2\pi C_\lambda^* C_{\lambda'} \hbar \alpha \delta(\lambda - \lambda') \end{aligned}$$

Therefore  $C_\lambda = \frac{1}{(2\pi \hbar \alpha)^{1/2}}$

② a) The potential energy  $V(x,y)$  is a quadratic form in terms of  $x$  and  $y$ . For the existence of discrete energy levels (i.e. bound states) it is necessary this form is positive-definite. In other words  $V(x,y)$  must have the shape of an elliptic paraboloid that opens in the positive direction of  $V$  values.

$$V(x,y) = \frac{mw^2}{2} \left[ \underbrace{x^2 + y^2 + 2\beta xy}_{\text{quadratic form}} \right] \quad \det \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} > 0$$

$$1 - \beta^2 > 0 \Rightarrow |\beta| < 1 \quad \leftarrow \text{condition for the existence of discrete energy levels}$$

b) To find the energy eigenvalues we can transform the Hamiltonian into two independent Hamiltonians - two uncoupled oscillators. Consider the rotation:

$$\begin{cases} X = ax + by \\ Y = -bx + ay \end{cases} \quad \begin{cases} x = aX - bY \\ y = bX + aY \end{cases}$$

$$a^2 + b^2 = 1$$

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x} \frac{\partial}{\partial X} + \frac{\partial Y}{\partial x} \frac{\partial}{\partial Y} = a \frac{\partial}{\partial X} - b \frac{\partial}{\partial Y} \Rightarrow p_x = a P_x - b P_y$$

$$\frac{\partial}{\partial y} = \frac{\partial X}{\partial y} \frac{\partial}{\partial X} + \frac{\partial Y}{\partial y} \frac{\partial}{\partial Y} = b \frac{\partial}{\partial X} + a \frac{\partial}{\partial Y} \Rightarrow p_y = b P_x + a P_y$$

Let us substitute that in the Hamiltonian

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{mw^2}{2} [x^2 + y^2 + 2\beta xy] = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{mw^2}{2} [X^2 + Y^2 + 2\beta ab(X^2 - Y^2)] = \underbrace{\frac{P_x^2}{2m} + \frac{mw^2}{2}(1+2\beta ab)X^2}_{H_X} + \underbrace{\frac{P_y^2}{2m} + \frac{mw^2}{2}(1-2\beta ab)Y^2}_{H_Y}$$

$$\text{If we denote } \omega_x = \omega \sqrt{1+2\beta ab} \quad \omega_y = \omega \sqrt{1-2\beta ab}$$

we can write the Hamiltonian as

$$H = \underbrace{\frac{P_x^2}{2m} + \frac{m\omega_x^2}{2} X^2}_{H_x} + \underbrace{\frac{P_y^2}{2m} + \frac{m\omega_y^2}{2} Y^2}_{H_y}$$

Since both  $H_x$  and  $H_y$  are independent and represent simple 1D oscillators we can find the total energy eigenvalues :

$$E = E_x + E_y = \hbar\omega_x(n_x + \frac{1}{2}) + \hbar\omega_y(n_y + \frac{1}{2})$$

In general, the energy levels are nondegenerate, unless  $\omega_x$  and  $\omega_y$  are commensurate.

(3) The linear combination of states  $|0\rangle$  and  $|1\rangle$ :

$$|\psi\rangle = a|0\rangle + b|1\rangle \quad \text{where } |a|^2 + |b|^2 = 1$$

$$\text{or } |b| = \sqrt{1 - |a|^2}$$

Both  $a$  and  $b$  are complex numbers and can be represented as

$$a = |a| e^{i\delta} \quad b = |b| e^{i\delta} = \sqrt{1 - |a|^2} e^{i\delta}$$

The expectation value of  $x$  is given by

$$\langle x \rangle = \langle \psi | x | \psi \rangle = |a|^2 \langle 0 | x | 0 \rangle + a^* b \langle 0 | x | 1 \rangle + b^* a \langle 1 | x | 0 \rangle + |b|^2 \langle 1 | x | 1 \rangle$$

Operator  $x$  in terms of  $a$  and  $a^\dagger$  is  $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$

Hence, the expectation value of  $x$  is then

$$\begin{aligned} \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \left[ |a|^2 \langle 0 | a + a^\dagger | 0 \rangle + a^* b \langle 0 | a + a^\dagger | 1 \rangle + b^* a \langle 1 | a + a^\dagger | 0 \rangle + \right. \\ &\quad \left. + |b|^2 \langle 1 | a + a^\dagger | 1 \rangle \right] \end{aligned}$$

using the fact that  $a|0\rangle = 0$ ,  $a^\dagger|0\rangle = |1\rangle$ ,  $a|1\rangle = |0\rangle$ ,  $a^\dagger|1\rangle = \sqrt{2}|2\rangle$  and  $\langle k | n \rangle = \delta_{kn}$ , we get

$$\begin{aligned} \langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}} [a^* b + b^* a] = \sqrt{\frac{\hbar}{2m\omega}} |a| \sqrt{1 - |a|^2} \left[ e^{-i\gamma + i\delta} + e^{-i\delta + i\gamma} \right] = \\ &= \sqrt{\frac{2\hbar}{m\omega}} |a| \sqrt{1 - |a|^2} \cos(\delta - \gamma) \end{aligned}$$

Now we just need to find the maximum of the above expression:  $\frac{\partial \langle x \rangle}{\partial |a|} = 0 \Rightarrow \sqrt{1 - |a|^2} - \frac{|a|^2}{\sqrt{1 - |a|^2}} = 0 \Rightarrow$

$$\Rightarrow 1 - |a|^2 - |a|^2 = 0 \Rightarrow |a| = \frac{1}{\sqrt{2}}$$

$$\frac{\partial \langle x \rangle}{\partial (\delta - \gamma)} = 0 \Rightarrow -\sin(\delta - \gamma) = 0 \Rightarrow \delta = \gamma + n\pi \quad n = 0, \pm 1, \pm 2$$

For a maximum we can pick  $\delta - \gamma = 0$ . Then

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \text{up to an arbitrary phase factor}$$

and

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

(4) a) Our unit vector  $\vec{n}$  is  $\vec{n} = \sin\gamma \hat{e}_x + \cos\gamma \hat{e}_z$

So the operator  $\vec{n} \cdot \vec{S}$  can be written as

$$\vec{n} \cdot \vec{S} = \sin\gamma \frac{\hbar}{2} \hat{S}_x + \cos\gamma \frac{\hbar}{2} \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}$$

The eigenvalues and eigenfunctions of this operator are

$$+\frac{\hbar}{2} : |\uparrow_{\vec{n}}\rangle = \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} \quad -\frac{\hbar}{2} : |\downarrow_{\vec{n}}\rangle = \begin{pmatrix} \sin\frac{\gamma}{2} \\ -\cos\frac{\gamma}{2} \end{pmatrix}$$

We can also find the eigenvalues and eigenfunctions of the operator  $S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . They are

$$+\frac{\hbar}{2} : |\uparrow_x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad -\frac{\hbar}{2} : |\downarrow_x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The probability of getting  $+\frac{\hbar}{2}$  when  $S_x$  is measured is given by

$$P = |\langle \uparrow_x | \uparrow_{\vec{n}} \rangle|^2 = \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} \right|^2 = \frac{1}{2} \left| \cos\frac{\gamma}{2} + \sin\frac{\gamma}{2} \right|^2 \\ = \frac{1}{2} (1 + 2 \cos\frac{\gamma}{2} \sin\frac{\gamma}{2}) = \frac{1}{2} (1 + \sin\gamma)$$

b)  $\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$

$$\langle S_x \rangle = \langle \uparrow_{\vec{n}} | S_x | \uparrow_{\vec{n}} \rangle = \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} \end{pmatrix} = \\ = \frac{\hbar}{2} 2 \cos\frac{\gamma}{2} \sin\frac{\gamma}{2} = \frac{\hbar}{2} \sin\gamma$$

$$\langle S_x^2 \rangle = \langle \uparrow_{\vec{n}} | S_x^2 | \uparrow_{\vec{n}} \rangle = \frac{\hbar^2}{4} \langle \uparrow_{\vec{n}} | 1 | \uparrow_{\vec{n}} \rangle = \frac{\hbar^2}{4}$$

Therefore

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - \frac{\hbar^2}{4} \sin^2\gamma} = \frac{\hbar}{2} |\cos\gamma|$$

⑤ a) To obtain the spin matrices for  $S=1$

we can consider the relations

$$S_+ |10\rangle = \hbar \sqrt{2} |11\rangle$$

$$S_- |11\rangle = \hbar \sqrt{2} |10\rangle$$

$$S_+ = S_x + iS_y$$

$$S_+ |1-1\rangle = \hbar \sqrt{2} |10\rangle$$

$$S_- |10\rangle = \hbar \sqrt{2} |1-1\rangle$$

$$S_- = S_x - iS_y$$

which lead to

$$S_+ = \hbar \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_- = \hbar \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z \text{ is obviously diagonal in its own basis: } S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

b) Our Hamiltonian is  $H = -\vec{\mu} \cdot \vec{B} = -gBS_x = -gB\hbar \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Its eigenvalues and eigenfunctions are

$$E_1 = gB\hbar \quad \Psi_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \quad E_2 = 0 \quad \Psi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$E_3 = -gB\hbar \quad \Psi_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

Since  $H$  is time-independent,  $\Psi(t)$  can be written as

$$\Psi(t) = C_1 \Psi_1 e^{-iE_1 t} + C_2 \Psi_2 e^{-iE_2 t} + C_3 \Psi_3 e^{-iE_3 t} =$$

$$= C_1 \Psi_1 e^{-igBt} + C_2 \Psi_2 + C_3 e^{+igBt}$$

$$\text{At } t=0 \quad \Psi(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} (\Psi_1 + \sqrt{2} \Psi_2 + \Psi_3)$$

From that we can find  $c_1, c_2, c_3$  : The final expression for  $\psi(t)$  is

$$\begin{aligned}\psi(t) &= \frac{1}{2} \psi_1 e^{-igBt} + \frac{1}{\sqrt{2}} \psi_2 + \frac{1}{2} \psi_3 e^{+igBt} \\ &= \begin{pmatrix} \frac{1}{2} (\cos gBt + 1) \\ -\frac{i}{\sqrt{2}} \sin gBt \\ \frac{1}{2} (\cos gBt - 1) \end{pmatrix}\end{aligned}$$

c)

$$\begin{aligned}P_{|1-1\rangle} &= |\langle 1-1 | \psi(t) \rangle|^2 = \left| \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} (\cos gBt + 1) \\ -\frac{i}{\sqrt{2}} \sin gBt \\ \frac{1}{2} (\cos gBt - 1) \end{pmatrix} \right|^2 = \\ &= \frac{1}{4} (1 - \cos gBt)^2 = \sin^2 \frac{gBt}{2}\end{aligned}$$

⑥ The Hamiltonian of a free particle is  $H = \frac{p^2}{2m}$

In the Heisenberg picture  $x(t) = U^{-1}x(0)U$ , where  $U$  is the time evolution operator. In our case

$$U = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{it}{2mh}p^2} = e^{-idp^2} \quad d \equiv \frac{t}{2mh}$$

The commutator we need to compute is then

$$[x(t), x(0)] = [U^{-1}x(0)U, x(0)] = [e^{idp^2}x(0)e^{-idp^2}, x(0)]$$

here  $p \equiv p(0)$ . We also know that  $x(0) = it \frac{\partial}{\partial p(0)}$

With that we can write the commutator as

$$[x(t), x(0)] = -\hbar^2 \left[ e^{idp^2} \frac{\partial}{\partial p} e^{-idp^2}, \frac{\partial}{\partial p} \right]$$

$$\begin{aligned} & \left[ e^{idp^2} \frac{\partial}{\partial p} e^{-idp^2}, \frac{\partial}{\partial p} \right] f(p) = \left( e^{idp^2} \frac{\partial}{\partial p} e^{-idp^2} f' - \frac{\partial}{\partial p} e^{idp^2} \frac{\partial}{\partial p} (e^{-idp^2} f') \right) = \\ & = \left( e^{idp^2} (-2idp e^{-idp^2} f' + e^{-idp^2} f'') - \frac{\partial}{\partial p} e^{idp^2} (-2idp e^{-idp^2} f + e^{-idp^2} f') \right) = \\ & = \left( -2idp f' + f'' - \frac{\partial}{\partial p} (-2idp f + f') \right) = \cancel{(-2idp f' + f'' + 2idp f + 2idp f' - f'')} \\ & = 2idf \end{aligned}$$

Hence, we can conclude that

$$[x(t), x(0)] = -\hbar^2 \cdot 2id = -\frac{i\hbar t}{m}$$