PHYS 451: Quantum Mechanics I - Spring 2017
Homework #3, due Tuesday February 7 in class

General properties of stationary states in 1D, quantum harmonic oscillator

1. Prove that for any 1D potential $V(x)$ that satisfies the following conditions
$$V(x) \to 0 \quad \text{for} \quad x \to \pm\infty,$$
and
$$\int\limits_{-\infty}^{\infty} V(x)dx < 0,$$
there exist at least one bound state with negative energy.
Hint: Use a wave function in the form $\varphi(x) \sim e^{-\alpha|x|}$ with $\alpha > 0$. This wave function can be expanded in terms of the eigenfunctions of the Hamiltonian. You can show that for small enough α the expectation value of the Hamiltonian with function $\varphi(x)$ is negative. Since the average energy for any state is always equal or greater than the ground state energy (show it, too) it proves the existence of the ground state with a negative energy.

2. Consider a charged harmonic oscillator in a uniform electric field $\mathcal{E}$. The potential energy of this system reads
$$V(x) = \frac{m\omega^2 x^2}{2} - e\mathcal{E}x.$$
   (a) Find the energy spectrum and the eigenfunctions for this system
   (b) Find the static polarizability of the system in the above eigenstates. The dipole moment μ, first (α), second (β), third (γ), and higher order polarizabilities of a system put in an external field are defined as the coefficients of the expansion of the energy in terms of the powers of $\mathcal{E}$:
$$E(\mathcal{E}) = E(0) - \mu(0)\mathcal{E} - \frac{\alpha(0)}{2!}\mathcal{E}^2 - \frac{\beta(0)}{3!}\mathcal{E}^3 - \ldots$$
   where
$$\mu(0) = -\left(\frac{\partial E}{\partial \mathcal{E}}\right)_{\mathcal{E}=0}, \quad \alpha(0) = -\left(\frac{\partial^2 E}{\partial \mathcal{E}^2}\right)_{\mathcal{E}=0}, \quad \beta(0) = -\left(\frac{\partial^3 E}{\partial \mathcal{E}^3}\right)_{\mathcal{E}=0}, \quad \ldots$$

3. A particle is in the ground state of a harmonic oscillator potential. What is the probability (give a numeric value) of finding the particle outside of classically allowed region? Do you expect the corresponding probability for a particle in state $n = 10$ (n is the quantum number) to be larger or smaller? Why?

4. Consider a particle of mass m in a harmonic oscillator potential, $V(x) = \frac{m\omega^2 x^2}{2}$. The initial state of the system is given by some function $f(x)$, i.e. $\Psi(x, t=0) = f(x)$. Show that the expectation value of x as a function of time can be written as $\langle x\rangle = A\cos(\lambda t) + B\sin(\lambda t)$. Find the expressions for A, B, and λ. There is a useful relation for the eigenfunctions of the harmonic oscillator that you may want to use:
$$\sqrt{\frac{m\omega}{\hbar}}\, x\psi_n(x) = \sqrt{\frac{n+1}{2}}\,\psi_{n+1} + \sqrt{\frac{n}{2}}\,\psi_{n-1}.$$