StudentID: ______________________

PHYS 451: Quantum Mechanics I – Spring 2017
Instructor: Sergiy Bubin
Midterm Exam 1

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Derive the virial theorem (i.e. the relation between $\langle T\rangle$ and $\langle V\rangle$) for the eigenstates of a system with the Hamiltonian

$$H = T + V = \frac{p^2}{2m} + \alpha x^{2n},$$

where $\alpha > 0$ and n is a positive integer.
Hint: it may be helpful to consider the time derivative of an expectation value of operator xp.

Problem 2. Consider the potential in the form of a step function:

$$V(x) = \begin{cases} 0, & x \leq 0, \\ V_0 & x > 0. \end{cases}$$

Given that the incident particles come from the left calculate the reflection coefficient for the case when $E > V_0$.

Problem 3. Using the formalism of the creation and annihilation operators, find the explicit matrix form of operators x, p, and $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$ in the basis of eigenstates of the harmonic oscillator.

Problem 4. Consider a three-level system with the Hamiltonian

$$H = \begin{pmatrix} a & 0 & ib \\ 0 & a & 0 \\ -ib & 0 & a \end{pmatrix},$$

where a and b are real positive constants. Find the projection operator that projects onto the subspace orthogonal to the ground state.

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\quad(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\,(2\alpha x^2 - 1)\,e^{-\alpha x^2/2}$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Dirac delta function

$\int\limits_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$ $\delta(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{ikx}dk$ $\delta(-x) = \delta(x)$ $\delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx$ $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$

or, in terms of $p = \hbar k$

$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx$ $f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{+\infty}\tilde{f}(p)e^{ipx/\hbar}dp$