

① The time evolution of the expectation value of operator x_p is described by the following equation

$$\frac{d}{dt} \langle x_p \rangle = \frac{i}{\hbar} \langle [H, x_p] \rangle + \underbrace{\left\langle \frac{\partial x_p}{\partial t} \right\rangle}_0$$

Now

$$[H, x_p] = \left[\frac{p^2}{2m} + \alpha x^{2n}, x_p \right] = \frac{1}{2m} [p^2, x_p] + \alpha [x^{2n}, x_p] =$$

$$= \frac{i\hbar^3}{2m} \left[\frac{d^2}{dx^2}, x \frac{d}{dx} \right] - i\hbar \alpha [x^{2n}, x \frac{d}{dx}]$$

$$\left[\frac{d^2}{dx^2}, x \frac{d}{dx} \right] = \frac{d^2}{dx^2} x \frac{d}{dx} - x \frac{d^3}{dx^3} = \frac{d}{dx} \left(\frac{d}{dx} + x \frac{d^2}{dx^2} \right) - x \frac{d^3}{dx^3}$$

$$= \frac{d^2}{dx^2} + \frac{d^2}{dx^2} + x \frac{d^3}{dx^3} - x \frac{d^3}{dx^3} = 2 \frac{d^2}{dx^2}$$

$$[x^{2n}, x \frac{d}{dx}] = x^{2n+1} \frac{d}{dx} - x \frac{d}{dx} x^{2n} = x^{2n+1} \frac{d}{dx} - x^{2n+1} \frac{d}{dx} - 2n x^{2n} = -2n x^{2n}$$

So

$$[H, x_p] = \frac{i\hbar^3}{2m} 2 \frac{d^2}{dx^2} + i\hbar \alpha 2n x^{2n} = -2i\hbar \frac{p^2}{2m} + 2i\hbar \alpha x^{2n}$$

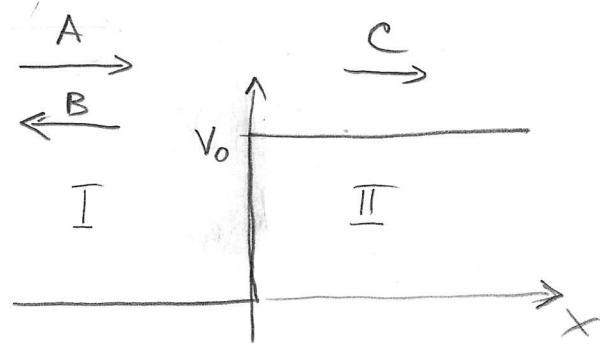
and

$$\frac{d}{dt} \langle x_p \rangle = 2 \left\langle -\frac{p^2}{2m} + \alpha x^{2n} \right\rangle$$

For stationary states $\frac{d}{dt} \langle x_p \rangle = 0$. This gives us the relation

$$\left\langle \frac{p^2}{2m} \right\rangle = \left\langle \alpha x^{2n} \right\rangle \quad \text{or} \quad \langle T \rangle = n \langle V \rangle$$

(2) Here we solve the Schrödinger equation in region I ($x < 0$) and II ($x > 0$) and then match the solutions Ψ_I and Ψ_{II} so that the total wave function ψ is continuous and its derivative is also continuous.



$$\Psi_I = Ae^{ikx} + Be^{-ikx} \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Psi_{II} = Ce^{ixc} \quad xc = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\Psi_I(0) = \Psi_{II}(0) \Rightarrow A + B = C$$

$$\Psi'_I(0) = \Psi'_{II}(0) \Rightarrow ik(A - B) = ix C$$

From the above two equations we can find

$$\frac{B}{A} = \frac{k - xc}{k + xc} \quad \frac{C}{A} = \frac{2k}{k + xc}$$

The transmission and reflection coefficients are then given by

$$T = \frac{|C|^2}{|A|^2} = \frac{4\sqrt{E(E-V_0)}}{(\sqrt{E} + \sqrt{E-V_0})^2} \quad R = \frac{|B|^2}{|A|^2} = \frac{(\sqrt{E} - \sqrt{E-V_0})^2}{(\sqrt{E} + \sqrt{E-V_0})^2}$$

$$\text{When } E \rightarrow V_0 \quad T \approx 4\sqrt{\frac{E-V_0}{V_0}} \sim \sqrt{E-V_0} \rightarrow 0 \quad R \rightarrow 1$$

$$\text{When } E \rightarrow \infty \quad R \approx \frac{V_0^2}{16E^2} \rightarrow 0 \quad T \rightarrow 1$$

(3)

$$\hat{a} = \sqrt{\frac{mc\omega}{2\hbar}} \hat{x} + \frac{i\hat{p}}{\sqrt{2mc\omega}}$$

$$\hat{a}^+ = \sqrt{\frac{mc\omega}{2\hbar}} \hat{x} - \frac{i\hat{p}}{\sqrt{2mc\omega}}$$

and

$$\hat{H} = \hbar\omega(a^+a + \frac{1}{2})$$

\hat{x} and \hat{p} in terms of \hat{a} and \hat{a}^+ are :

$$\hat{x} = \sqrt{\frac{\hbar}{2mc\omega}} (a^+ + a)$$

$$\hat{p} = i\sqrt{\frac{mc\omega}{2}} (a^+ - a)$$

We also know that

$$|nh\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad a^+|nh\rangle = \sqrt{n+1}|n+1\rangle \quad \text{where} \quad |n\rangle \equiv |\Psi_n\rangle$$

With that matrix elements of \hat{x} are :

$$\langle n|\hat{x}|k\rangle = \sqrt{\frac{\hbar}{2mc\omega}} \langle n|(a^+ + a)|k\rangle$$

Since both a or a^+ when acting on $|n\rangle$ produce an eigenstate of \hat{H} (which are orthogonal to all other eigenstates) we can immediately write

$$\begin{aligned} \langle n|\hat{x}|k\rangle &= \sqrt{\frac{\hbar}{2mc\omega}} \left[\sqrt{k+1}\langle n|k+1\rangle + \sqrt{k}\langle n|k-1\rangle \right] = \sqrt{\frac{\hbar}{2mc\omega}} \left[\sqrt{k+1}\delta_{n,k+1} + \sqrt{k}\delta_{n,k-1} \right] \\ &= \sqrt{\frac{\hbar}{2mc\omega}} \left[\sqrt{k+1}\delta_{n,k+1} + \sqrt{n+1}\delta_{n+1,k} \right] \end{aligned}$$

In matrix form this looks as follow (remember that the index begins with 0 for our case) :

$$\hat{x} = \sqrt{\frac{\hbar}{2mc\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \dots & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Similarly, for \hat{p} we have $\langle n|\hat{p}|k\rangle = i\sqrt{\frac{mc\omega}{2}} \langle n|(a^+ - a)|k\rangle = i\sqrt{\frac{mc\omega}{2}} (\sqrt{k+1}\langle n|k+1\rangle - \sqrt{k}\langle n|k-1\rangle) = i\sqrt{\frac{mc\omega}{2}} [\sqrt{k+1}\delta_{n,k+1} - \sqrt{k+1}\delta_{n+1,k}]$

$$\hat{P} = \sqrt{\frac{\hbar \omega}{2}} \begin{pmatrix} 0 & -i\sqrt{1} & 0 & 0 & \dots \\ i\sqrt{1} & 0 & -i\sqrt{2} & 0 & \dots \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & \dots \\ 0 & 0 & i\sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Lastly, for \hat{H} we have:

$$\langle n | \hat{H} | n \rangle = E_n \langle n | n \rangle = E_n \delta_{nn} = \hbar \omega (n + \frac{1}{2}) \delta_{nn}$$

$$\hat{H} = \hbar \omega \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & \frac{3}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{5}{2} & 0 & \dots \\ 0 & 0 & 0 & \frac{7}{2} & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

The matrix of any operator is diagonal when computed in the basis of this operator's eigenstates (with eigenvalues on the diagonal)

④ First we need to find the ground state of this Hamiltonian:

$$\begin{vmatrix} a-\lambda & 0 & ib \\ 0 & a-\lambda & 0 \\ -ib & 0 & a-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)^3 - b^2(a-\lambda) = 0$$

so $\lambda_1 = a-b$ $\lambda_2 = a$ $\lambda_3 = a+b$

Obviously λ_1 is the lowest eigenvalue because both a and b are positive. The corresponding eigenvector $|v_1\rangle$

$$\begin{pmatrix} b & 0 & ib \\ 0 & b & 0 \\ -ib & 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x + iz = 0 \\ y = 0 \\ -ibx + z = 0 \end{array} \Rightarrow |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$$

The projection operator on $|v_1\rangle$ is $P_1 = |v_1\rangle\langle v_1|$
 The projection operator on the subspace orthogonal to $|v_1\rangle$ is $P_{\perp} = 1 - P_1$

So

$$P_{\perp} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} i & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{i}{2} \\ 0 & 1 & 0 \\ -\frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}$$