

① a) Since  $H_0$  commutes with  $L_z$ , the eigenstates of  $H_0$  and  $L_z$  are also eigenstates of  $H$  with modified eigenvalues

$$E_{nm} = E_n^{(0)} - m\hbar\omega \quad (\text{note the dependence on } m)$$

b) Hamiltonian  $H$  does not depend on time explicitly, so

$$|\psi(t)\rangle = \sum'_{n,e,m} c_{nem} e^{\frac{-iE_{nm}t}{\hbar}} |nem\rangle$$

In our case  $|\psi(0)\rangle = \frac{1}{\sqrt{2}}|21-1\rangle - \frac{1}{\sqrt{2}}|211\rangle$ . Therefore,

$$\begin{aligned} \Psi(+)&= \frac{1}{\sqrt{2}} e^{\frac{-iE_{2-1}t}{\hbar}} |21-1\rangle - \frac{1}{\sqrt{2}} e^{\frac{-iE_{21}t}{\hbar}} |211\rangle = \\ &= \frac{1}{\sqrt{2}} e^{\frac{-iE_2^{(0)}t}{\hbar}} \left( e^{i\omega t} |21-1\rangle - e^{-i\omega t} |211\rangle \right) \end{aligned}$$

The probabilities of finding the system in states  $|2px\rangle$ ,  $|2py\rangle$ , and  $|2pz\rangle$  are:

$$\begin{aligned} P_{2px} &= |\langle 2px | \Psi(+)\rangle|^2 = \frac{1}{4} \left| (\langle 21-1 | - \langle 211 |) (e^{i\omega t} |21-1\rangle - e^{-i\omega t} |211\rangle) \right|^2 \\ &= \frac{1}{4} |2 \cos \omega t|^2 = \cos^2 \omega t \end{aligned}$$

$$\begin{aligned} P_{2py} &= |\langle 2py | \Psi(+)\rangle|^2 = \frac{1}{4} \left| (\langle 21-1 | + \langle 211 |) (e^{i\omega t} |21-1\rangle - e^{-i\omega t} |211\rangle) \right|^2 \\ &= \frac{1}{4} |2i \sin \omega t|^2 = \sin^2 \omega t \end{aligned}$$

$$P_{2pz} = |\langle 2pz | \Psi(+)\rangle|^2 = \frac{1}{2} |\langle 210 | (e^{i\omega t} |21-1\rangle - e^{-i\omega t} |211\rangle)|^2 = 0$$

② Let us find eigenstates of operator  $\vec{n} \cdot \vec{S}$  corresponding to the projection of spin  $\vec{S}$  on an arbitrary axis  $\vec{n}$ :  $\vec{n} = \sin\theta \cos\phi \hat{e}_x + \sin\theta \sin\phi \hat{e}_y + \cos\theta \hat{e}_z$

$$\begin{aligned}\vec{n} \cdot \vec{S} &= \frac{\hbar}{2} \left[ \sin\theta \cos\phi S_x + \sin\theta \sin\phi S_y + \cos\theta S_z \right] = \\ &= \frac{\hbar}{2} \left[ \sin\theta \cos\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta \sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \\ &= \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\end{aligned}$$

The eigenvalues of this matrix are, obviously,  $\pm \frac{\hbar}{2}$ .  
The eigenvectors are given by

$$+ \frac{\hbar}{2}: |\chi_{\vec{n}\uparrow}\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix} \quad - \frac{\hbar}{2}: |\chi_{\vec{n}\downarrow}\rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

In our case

$$\begin{pmatrix} a \\ b \end{pmatrix} = |\chi_{\vec{n}\downarrow}\rangle = \begin{pmatrix} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

Therefore we can relate  $a$  and  $b$  to  $\phi$  and  $\theta$  as follows:

$$\begin{aligned}a &= \sin \frac{\theta}{2} \quad b = -\cos \frac{\theta}{2} e^{i\phi} \quad \text{or} \quad \theta = 2 \arcsin a \\ \phi &= \ln \left[ \frac{\cos(\arcsin a)}{b} \right] \\ &= \ln \frac{\sqrt{1-a^2}}{b}\end{aligned}$$

(3) Let us denote the basis states  $|sm\rangle$ , where  $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$  (these states are eigenstates of both  $\hat{S}^2$  and  $\hat{S}_z$ ) as follows

$$|\frac{3}{2}, \frac{3}{2}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\frac{3}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\frac{3}{2}, -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\frac{3}{2}, -\frac{3}{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Now let us consider the ladder operators  $S_+$  and  $S_-$ . Their action on states  $|sm\rangle$  is

$$S_{\pm} |\frac{3}{2}, m\rangle = \hbar \sqrt{\frac{15}{4} - m(m \pm 1)} |\frac{3}{2}, m \pm 1\rangle$$

More specifically,

$$S_+ |\frac{3}{2}, \frac{3}{2}\rangle = 0$$

$$S_- |\frac{3}{2}, \frac{3}{2}\rangle = \sqrt{3} \hbar |\frac{3}{2}, \frac{1}{2}\rangle$$

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$$S_- |\frac{3}{2}, \frac{1}{2}\rangle = 2 \hbar |\frac{3}{2}, -\frac{1}{2}\rangle$$

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$$S_- |\frac{3}{2}, -\frac{1}{2}\rangle = \sqrt{3} \hbar |\frac{3}{2}, -\frac{3}{2}\rangle$$

$$S_+ |\frac{3}{2}, -\frac{3}{2}\rangle = \sqrt{3} \hbar |\frac{3}{2}, -\frac{1}{2}\rangle$$

$$S_- |\frac{3}{2}, -\frac{3}{2}\rangle = 0$$

In matrix form it looks as follows:

$$S_+ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_- \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix}$$

which gives us

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Knowing  $S_+$  and  $S_-$  we can compute  $S_x$  and  $S_y$ :

$$S_x = \frac{1}{2} (S_+ + S_-) = \hbar \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i} (S_+ - S_-) = \hbar \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}i & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{pmatrix}$$

The eigenvalues of both  $S_x$  and  $S_y$  are

$+\frac{3}{2}\hbar, +\frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar$  (Same as the eigenvalues of  $S_z$ )

The latter can be deduced based on isotropy -  
the y- or x-direction is no different than the  
z-direction.

(4) We need to compute  $\langle \psi | (x_1 - x_2)^2 | \psi \rangle =$   
 $= \langle \psi | x_1^2 | \psi \rangle + \langle \psi | x_2^2 | \psi \rangle - 2 \langle \psi | x_1 x_2 | \psi \rangle$

a)  $|\psi_d\rangle = \phi_0(x_1) \phi_1(x_2)$

$$\langle (x_1 - x_2)^2 \rangle_d = \langle 0 | x^2 | 0 \rangle \langle 1 | 1 \rangle + \langle 0 | 0 \rangle \langle 1 | x^2 | 1 \rangle - 2 \langle 0 | x | 0 \rangle \langle 1 | x | 1 \rangle$$

Matrix elements  $\langle n | x | n \rangle$  and  $\langle n | x^2 | n \rangle$  can be found in the formula sheet. We obtain

$$\langle (x_1 - x_2)^2 \rangle_d = \frac{t}{2m\omega} \cdot 1 + 1 \cdot \frac{3t}{2m\omega} - 2 \cdot 0 \cdot 0 = \frac{2t}{m\omega}$$

b,c)  $|\psi_{bf}\rangle = \frac{1}{\sqrt{2}} [\phi_0(x_1) \phi_1(x_2) \pm \phi_1(x_1) \phi_0(x_2)]$

$$\langle (x_1 - x_2)^2 \rangle_{bf} = \frac{1}{2} [\langle 0 | x^2 | 0 \rangle \langle 1 | 1 \rangle + \langle 0 | 0 \rangle \langle 1 | x^2 | 1 \rangle - 2 \langle 0 | x | 0 \rangle \langle 1 | x | 1 \rangle]$$

$$+ \langle 1 | x^2 | 1 \rangle \langle 0 | 0 \rangle + \langle 1 | 1 \rangle \langle 0 | x^2 | 0 \rangle - 2 \langle 1 | x | 1 \rangle \langle 0 | x | 0 \rangle$$

$$\pm \langle 1 | x^2 | 0 \rangle \langle 0 | 1 \rangle \pm \langle 1 | 0 \rangle \langle 0 | x^2 | 1 \rangle \mp 2 \langle 1 | x | 0 \rangle \langle 1 | x | 0 \rangle]$$

$$\pm \langle 0 | x^2 | 1 \rangle \langle 1 | 0 \rangle \pm \langle 0 | 1 \rangle \langle 1 | x^2 | 0 \rangle \mp 2 \langle 0 | x | 1 \rangle \langle 0 | x | 1 \rangle]$$

$$= \frac{2t}{m\omega} \mp 2 |\langle 1 | x | 0 \rangle|^2 = \frac{2t}{m\omega} \mp 2 \left| \sqrt{\frac{t}{2m\omega}} \right|^2 = \frac{2t}{m\omega} \mp \frac{t}{m\omega}$$

so for bosons  $\langle (x_1 - x_2)^2 \rangle_b = \frac{t}{m\omega}$

for fermions  $\langle (x_1 - x_2)^2 \rangle_f = \frac{3t}{m\omega}$