PHYS 451 Quantum Mechanics I (Spring 2018)
Homework #3, due Thursday Feb 8 in class

Harmonic oscillator, Fourier transform, Wavepackets, Dirac delta function

1. A particle is in the ground state of a harmonic oscillator potential. What is the probability (give a numeric value) of finding the particle outside of classically allowed region? Do you expect the corresponding probability for a particle in the state $n = 10$ (n is the quantum number) to be larger or smaller? Why?

2. Consider a free particle in 1D with a wave function

$$\psi(x) = \begin{cases} Ae^{ik_0x}, & -a/2 \le x \le a/2 \\ 0, & |x| > a/2 \end{cases},$$

   where A, a, and k_0 are positive constants.

   (a) Find A and sketch $\mathrm{Re}[\psi(x)]$, $\mathrm{Im}[\psi(x)]$, and $|\psi(x)|^2$.

   (b) Find the momentum space wave function, $\tilde{\psi}(k)$. Show that it is normalized.

   (c) Find the uncertainties of the position and momentum, Δx and δp. Is there is anything wrong with that? If so, explain the origin of the trouble.

3. Consider a free particle of mass m in 1D that is initially very strongly localized at the origin ($x = 0$) and has the following wave function:

$$\Psi(x, t{=}0) \; = \; Ce^{-\alpha x^2/2}e^{ik_0x},$$

   where C, α, and k_0 are some positive constants, and α has a very large value.

   (a) Find C.

   (b) Show that the initial probability density approaches the delta function when $\alpha \to \infty$.

   (c) Keeping α finite determine the evolved wave function $\Psi(x, t)$ at $t > 0$.

   (d) Write down the position probability density at $t > 0$ and take the limit $\alpha \to \infty$. If you do so you will observe that even for very small values of time ($t \sim \frac{m}{\hbar\alpha}$) it becomes space independent. Give a physical interpretation of this behaviour (hint: think about the momentum distribution in the initial wave packet).

4. Show that

$$g(x) = \frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2}$$

   can be a valid representation of the Dirac delta function in the limit $\epsilon \to 0^+$. Namely, show that in this limit

   (a) $\int\limits_{-\infty}^{+\infty} g(x)f(x)dx = f(0)$ for any reasonably "nice" function $f(x)$.

   (b) $g(x) = g(-x)$.

   (c) $xg(x) = 0$.

   (d) $g(cx) = \frac{1}{|c|}g(x)$.

   (e) $g'(-x) = -g'(x)$.

   (f) $xg'(x) = -g(x)$.