PHYS 451 Quantum Mechanics I (Spring 2018)
Homework #5, due Thursday Feb 22 in class

Commutators, Operators, Hermicity

1. Determine if the following operators are linear, hermitian, nonhermitian, or antihermitian (an operator $\hat{O}$ is called antihermitian if $\hat{O}^\dagger = -\hat{O}$):
   (a) Scaling operator: $\hat{S}_\alpha f(x) = \sqrt{\alpha} f(\alpha x) \quad (\alpha > 0)$
   (b) $\hat{x}\hat{p}_x$
   (c) $i(\hat{A}\hat{B} - \hat{B}\hat{A})$, if it is known that $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$
   (d) $\hat{C} - \hat{C}^\dagger$
   (e) $\alpha\hat{x} - \beta\frac{d}{dx} \quad (\alpha, \beta > 0)$
   (f) Projection operator on some given state $|\alpha\rangle$

2. Prove the following operator relation: $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$

3. The action of the translation operator $\hat{T}_a$ on an arbitrary function $f(x)$ is given by
$$\hat{T}_a f(x) = f(x + a).$$
   Essentially the translation operator $\hat{T}_a$ shifts (or translates) the argument of a function by a. Find the following commutators:
   (a) $[\hat{x}, \hat{T}_a]$
   (b) $[\hat{p}, \hat{T}_a]$

   where $\hat{x}$ and $\hat{p}$ are the position and momentum operators respectively. It might be helpful to recall that
$$f(x + a) = f(x) + f'(x)a + \frac{1}{2!}f''(x)a^2 + \ldots$$
   and that
$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad \text{(in the position space)}$$
$$\hat{x} = i\hbar\frac{\partial}{\partial p} \qquad \text{(in the momentum space).}$$
   If it is unclear where the latter expression for $\hat{x}$ comes from, you can easily deduce it if you think about the Fourier transform of function $x\psi(x)$ and how it is related to the Fourier transform of $\psi(x)$.

4. Assuming that λ is a small parameter, find an expansion of the operator $(\hat{A} - \lambda\hat{B})^{-1}$ in the powers of λ.

5. A unitary operator is such an operator that satisfies the relation
$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{1} \quad \text{or, equivalently,} \quad U^{-1} = U^\dagger.$$
   In a simple language, a unitary operator preserves the "length" and "angles" between vectors, and it can be considered as a kind of rotation operator in complex vector space.
   (a) Show that the eigenvalues of a unitary operator λ_i are equal to unity, i.e. $\lambda_k = e^{i\alpha_k}$, where α_k is a real number.
   (b) Show that the product of two unitary operators U and V is also a unitary operator.
   (c) Can an operator be both Hermitian and Unitary?
   (d) If $\hat{A}$ is a Hermitian operator, show that $e^{i\hat{A}}$ is unitary.