Orbital angular momentum

To treat problems with rotational symmetry, it is useful to introduce the orbital angular momentum operator \( \hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \):

\[
\hat{\mathbf{L}} = \begin{vmatrix}
\hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\
\hat{x} & \hat{y} & \hat{z} \\
\hat{p}_x & \hat{p}_y & \hat{p}_z 
\end{vmatrix}
\]

or in components

\[
\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x
\]

\( \hat{\mathbf{L}} \) is a Hermitian operator, i.e. \( \hat{\mathbf{L}}^\dagger = \hat{\mathbf{L}} \), which can be easily verified. For example

\[
\hat{L}_z^\dagger = \hat{\mathbf{p}}_y \hat{x} - \hat{\mathbf{p}}_x \hat{y} = \hat{\mathbf{p}}_y \hat{x} - \hat{\mathbf{p}}_x \hat{y} = \hat{L}_z
\]

In the above check we used the fact that

\[
[\hat{\mathbf{r}}_i, \hat{\mathbf{p}}_j] = i\hbar \delta_{ij} \quad i, j = 1, 2, 3
\]

\[
\hat{\mathbf{r}} = \begin{pmatrix} \hat{r}_x \\ \hat{r}_y \\ \hat{r}_z \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \quad \hat{\mathbf{p}} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{p}}_x \\ \hat{\mathbf{p}}_y \\ \hat{\mathbf{p}}_z \end{pmatrix}
\]

For simplicity, we will no longer use hats to denote operators unless there is ambiguity. In most cases it is obvious when we deal with operators.

An essential property of the angular momentum operator is that its components do not commute with one another:
\[ \begin{align*}
[L_x, L_y] &= \left[ y p_z - z p_y, z p_x - x p_z \right] = [y p_z, z p_x] + [z p_y, x p_z] \\
&= \sqrt{y} [p_z, z] p_x + x [z, p_z] p_y = i\hbar (x p_y - y p_x) = i\hbar L_z
\end{align*} \]

Similarly, \[ [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y \]

This can also be written as
\[ [L_i, L_j] = i\hbar \varepsilon_{ijk} L_k \]

where \( \varepsilon_{ijk} \) is the antisymmetric tensor:
\[ \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1 \]
\[ \varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj} \]

\( \varepsilon_{ijk} \) is known as the Levi-Civita symbol.

Another compact way to express the commutation relations between \( L_i \) and \( L_j \) is
\[ [\mathbf{L} \times \mathbf{L}] = i\hbar \mathbf{L} \]

Note that the vector (cross) product would be zero if \( \mathbf{L} \) were ordinary vectors (not operators).

Since \( L_x, L_y, \) and \( L_z \) do not commute, they correspond to incompatible observables. The above commutation relations imply the uncertainty relations
\[ \Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle| \quad \text{(recall } \Delta A \Delta B \geq \frac{1}{2} \langle [A, B] \rangle) \]

where \( \Delta \) as usual, denotes the standard deviation.

Therefore, it is impossible to know simultaneously two different components of \( \mathbf{L} \) (unless \( \langle \mathbf{L} \rangle = 0 \)).
However, one may notice that each of the components of $\mathbf{L}$ commutes with $L^2$:

$$\left[ L^2, L_i \right] = 0 \quad \Rightarrow \quad L^2 = L_x^2 + L_y^2 + L_z^2$$

For example,

$$\left[ L^2, L_x \right] = \left[ L_x^2, L_x \right] + \left[ L_y^2, L_x \right] + \left[ L_z^2, L_x \right] = 0$$

$$= L_y \left[ L_y, L_x \right] + \left[ L_y, L_x \right] L_y + L_z \left[ L_z, L_x \right] + \left[ L_z, L_x \right] L_z$$

$$= -i \hbar L_y - i \hbar L_y + i \hbar L_z + i \hbar L_z$$

$$= 0$$

Therefore, $L^2$ is compatible with each component of $\mathbf{L}$. We can find simultaneous eigenstates of $L^2$ and (say) $L_z$:

$$L^2 f = \lambda f \quad L_z f = \mu f$$

The ladder operator method for angular momentum

Define two new operators:

$$L_+ = L_x + i L_y \quad L_- = L_x - i L_y = L_+$$

Clearly

$$\left[ L^2, L_\pm \right] = 0$$

$$\left[ L_z, L_+ \right] = \left[ L_z, L_x + i L_y \right] = i \hbar L_y + i \left( -i \hbar L_x \right) = \hbar (L_x + i L_y) = \hbar L_+$$

$$\left[ L_z, L_- \right] = i \hbar L_y - i \hbar L_x = -i \hbar L_-$$

$$\left[ L_+, L_- \right] = \left[ L_x + i L_y, L_x - i L_y \right] = -i \left[ L_x, L_y \right] + i \left[ L_y, L_x \right] = 2 \hbar L_z$$
Furthermore, \( L_+ L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + hL_z \)
\[ L_- L_+ = L_x^2 + L_y^2 + i[L_x, L_y] = L_x^2 + L_y^2 - hL_z \]

Thus we can write \( L^2 \) as
\[ L^2 = \frac{1}{2}(L_+ L_- + L_- L_+) + L_z \]

From the commutator \( [L_z, L_+] = hL_+ \), we have:
\[ L_z L_+ = L_+(L_z + h) \]

When acting on \( \phi \) (eigenfunction), it yields:
\[ L_+ L_+ \phi_{\lambda \mu} = L_+(L_z + h)\phi_{\lambda \mu} = (\mu + h)\phi_{\lambda \mu} \]

\[ \Rightarrow L_+ \phi_{\lambda \mu} \text{ is an eigenstate of } L_z \text{ with eigenvalue } (\mu + h). \]

Similarly,
\[ L_z L_- \phi_{\lambda \mu} = (\mu - h)\phi_{\lambda \mu} \]

\( L_+ \) and \( L_- \) are known as the raising and lowering operators.

For a given value of \( \lambda \), we obtain a ladder of states, whose eigenvalues are evenly separated by \( h \). To ascend the ladder, we apply \( L_+ \), to descend we apply \( L_- \). This process, however, cannot go forever. Eventually we will reach a state for which the \( z \)-component exceeds the total \( \lambda \).
Hence the must be some maximum such that
\[ L f_{\text{top}} = 0 \]
Now, let \( \lambda \) be this maximum \( L_2 \) eigenvalue:
\[ L_2 f_{\text{top}} = \lambda f_{\text{top}} \]
Using the fact that \( L^2 = L_+ L_- + L_2 + t L_2 \)
we obtain that
\[ L^2 f_{\text{top}} = (L_- L_+ L_2^2 + t L_2) f_{\text{top}} = (0 + t^2 e^2 + t^2 e^f) f_{\text{top}} = t^2 e (l+1) f_{\text{top}} \]
Thus,
\[ \lambda = t^2 e (l+1) \]
This defines the eigenvalue of \( L^2 \) in terms of the maximum eigenvalue of \( L_2 \).

Similarly, there is a bottom limit \( f_{\text{bot}} \) such that
\[ L_- f_{\text{bot}} = 0 \]
\[ L_2 f_{\text{bot}} = \lambda f_{\text{bot}} \]
\[ L^2 f_{\text{bot}} = (L_+ L_- + L_2^2 - t L_2) f_{\text{bot}} = (0 + t^2 e'^2 - t^2 e) f_{\text{bot}} = t^2 e' (l'-1) f_{\text{bot}} \]
and
\[ \lambda = t^2 e' (l'-1) \]
We now have \( e(l+1) = e' (l'-1) \)
Since \( e' \) cannot be equal to \( l+1 \), the only solution is
\[ e' = - e \]
The eigenvalues of $L_2$ are $\mu = \ell \hbar$, where $\mu$ runs from $-\ell$ to $+\ell$ in $N$ integer steps.

$$\ell = -\ell + N$$

Hence $\ell = N/2$ and $\ell$ must be either an integer or half-integer.

The eigenfunctions are characterized by the numbers $\ell$ and $m$:

$$L^2 \psi_{\ell m} = \hbar^2 \ell (\ell + 1) \psi_{\ell m}, \quad L^2 \psi_{\ell m} = \hbar^2 m^2 \psi_{\ell m}$$

where $\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$.

There are $2\ell + 1$ different $m$'s for each given $\ell$. 