PHYS 451: Quantum Mechanics I – Spring 2018
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Midterm Exam 1

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.

- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.

- No communication with classmates is allowed during the exam.

- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).

- Write legibly. If I cannot read and understand it then I will not be able to grade it.

- Make sure pages are stapled together before submitting your work.

Problem 1. Consider a particle of mass m moving in a 1D potential of the form

$$V(x) = \gamma|x|,$$

where γ is a positive constant. The exact analytic solutions, E_n and $\psi_n(x)$, where n is the quantum number labelling states, to the corresponding Schrödinger equation are a little difficult to obtain (although, in principle, that can be done). However, things become easier in the limit when $n \to \infty$. What are the expectation values of x and x^2 in this limit? Express your result through E_n (consider it known) and any relevant constants that appear in the Hamiltonian (i.e. $\hbar$, m, γ).

Problem 2. The wave function of a particle of mass m moving in 1D $(-\infty \le x \le +\infty)$ is given by

$$\psi(x) = A(\beta - x^2)e^{-\frac{x^2}{2\beta}},$$

where A and β are some positive constants. Now if the particle's momentum is measured, what value will occur most often?

Problem 3.

(a) The translation operator, $\hat{T}_a$, is an operator that displaces a quantum mechanical system by a finite distance a. As you may know, its explicit form is given by $\hat{T}_a = e^{-i\hat{p}a/\hbar}$. What is the adjoint operator, $\hat{T}_a^\dagger$? Knowing the latter, compute how an arbitrary potential $V(x)$ is changed under the transformation $\hat{T}_a^\dagger V(x)\hat{T}_a$.

(b) Compute the commutator $[\hat{T}_a, V(x)]$

(c) Now assume that the Hamiltonian of a system is invariant under arbitrary translations. Show that this necessarily leads to the conservation of momentum, (i.e. the expectation value of the momentum operator remains constant).

Problem 4. Consider a hydrogen-like atom with $Z = 2$ (Z defines the nuclear charge) in the ground state.

(a) What is the most probable value of r – the distance between the nucleus and the electron?

(b) Let us assume that the nucleus is not a point particle but a tiny sphere of radius b, such that $b \ll a_0$, where a_0 is the Bohr radius. What is the probability of finding the electron inside the nucleus? If you wish you can make any reasonable approximation here.

(c) Suddenly, at $t = 0$, something happens to the nucleus and it instantaneously looses it charge so that Z becomes 1 (you can think of some abstract decay with an emission of a positively charged proton by the nucleus). Yet the electron continues to orbit around the new nucleus that has $Z = 1$. What is the probability that the atom remains in the ground state at $t > 0$?

Appendix: formula sheet

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p, \;\; \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k}, \;\; E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q
(generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \;\; n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\; n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\; n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}e^{-\alpha x^2/2}, \;\; \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}x\,e^{-\alpha x^2/2}, \;\; \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\left(2\alpha x^2 - 1\right)e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{n}\,\delta_{k,n-1}\right)$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom ($a = \frac{a_0}{Z} = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$E_n = -\frac{m}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} = -\frac{\hbar^2}{2ma^2},$

The few first radial wave functions R_{nl} for the hydrogen atom

$$R_{10} = 2a^{-3/2}\, e^{-\frac{r}{a}} \qquad R_{20} = \frac{1}{\sqrt{2}} a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right) e^{-\frac{r}{2a}} \qquad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2}\, \frac{r}{a}\, e^{-\frac{r}{2a}}$$

The few first spherical harmonics

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\frac{x\pm iy}{r}$$

Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} \tilde{f}(p)e^{ipx/\hbar}dp$$

Useful integrals

$$\int_0^{\infty} x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\operatorname{Re}\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^{\infty} x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\operatorname{Re}\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^{\infty} x^{k}e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\operatorname{Re}\gamma > 0,\ k = 0,1,2,...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\operatorname{Re}\beta > 0)$$

$$\int_0^{\pi} \sin^{2k} x\, dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int_0^{\pi} \sin^{2k+1} x\, dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$