Problem 1. Consider a 1D quantum harmonic oscillator of frequency ω, initially prepared in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$, where $|n\rangle$ stands for an egenstate corresponding to quantum number n.

(a) Is this harmonic oscillator in a stationary state?

(b) Will the Heisenberg uncertainty principle hold at any later moment of time?

Do not just answer questions. Make sure to prove, show, or explain your point.

Solution:

(a) Because the potential $(m\omega^2x^2/2)$ does not depend of time, the time-dependent wave function of the system is given by

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2}}\phi_0(x)e^{-iE_0t/\hbar} + \frac{1}{\sqrt{2}}\phi_1(x)e^{-iE_1t/\hbar} \\ &= \frac{1}{\sqrt{2}}e^{-i\omega t/2}\Big(\phi_0(x) + \phi_1(x)e^{-i\omega t}\Big),\end{aligned}$$

where $E_n = \hbar\omega(n+1/2)$, and $\phi_n(x) = |n\rangle$ are the energy eigenvalues and eigenfunctions of the Hamiltonian, respectively. Note that $\phi_n(x)$ are real functions. In a stationary state, the expectation values of physical observables and particle density are supposed to be time-independent (as the term "stationary" suggests). In our case the density is

$$\begin{aligned}\rho = |\Psi(x,t)|^2 &= \frac{1}{2}\Big(\phi_0^2(x) + \phi_1^2(x) + \phi_0(x)\phi_1(x)e^{-i\omega t} + \phi_1(x)\phi_0(x)e^{i\omega t}\Big) \\ &= \frac{1}{2}\Big(\phi_0^2(x) + \phi_1^2(x) + 2\phi_0(x)\phi_1(x)\cos\omega t\Big).\end{aligned}$$

Clearly, ρ depends on time. Therefore, our harmonic oscillator, inititally in state $|\psi\rangle$, is NOT in a stationary state.

(b) In order to see whether the Heisenberg uncertainty principle holds or not, we need to compute the product $\Delta x\Delta p = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}\sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2}$. Let us do it, keeping in mind that $|n\rangle$ are real functions in the coordinate representation and $\hat{p}$ is a hermitian operator, for which $\langle k|\hat{p}|n\rangle = \langle n|\hat{p}|k\rangle^* = -\langle n|\hat{p}|k\rangle$:

$$\langle\Psi|\hat{x}|\Psi\rangle = \frac{1}{2}\Big(\langle 0|\hat{x}|0\rangle + \langle 1|\hat{x}|1\rangle + 2\langle 1|\hat{x}|0\rangle\cos\omega t\Big)\,,$$

$$\langle\Psi|\hat{x}^2|\Psi\rangle = \frac{1}{2}\Big(\langle 0|\hat{x}^2|0\rangle + \langle 1|\hat{x}^2|1\rangle + 2\langle 1|\hat{x}^2|0\rangle\cos\omega t\Big)\,,$$

$$\langle\Psi|\hat{p}|\Psi\rangle = \frac{1}{2}\Big(\langle 0|\hat{p}|0\rangle + \langle 1|\hat{p}|1\rangle + 2i\langle 1|\hat{p}|0\rangle\sin\omega t\Big)\,,$$

$$\langle\Psi|\hat{p}^2|\Psi\rangle = \frac{1}{2}\Big(\langle 0|\hat{p}^2|0\rangle + \langle 1|\hat{p}^2|1\rangle + 2\langle 1|\hat{p}^2|0\rangle\cos\omega t\Big)\,.$$

The matrix elements of $\hat{x}$, $\hat{x}^2$, $\hat{p}$, and $\hat{p}^2$ in the harmonic oscillator basis are known and can be taken from textbooks/notes (otherwise the integrals with ϕ_0 and ϕ_1 can be easily evaluated using their explicit form):

$$\langle n|\hat{x}|k\rangle = \sqrt{\frac{1}{2\alpha}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}\right)\,,$$

$$\langle n|\hat{x}^2|k\rangle = \frac{1}{2\alpha}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)\,,$$

$$\langle n|\hat{p}|k\rangle = -i\hbar\sqrt{\frac{\alpha}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{k+1}\,\delta_{n,k+1}\right)\,,$$

$$\langle n|\hat{p}^2|k\rangle = -\hbar^2\frac{\alpha}{2}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} - (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)\,,$$

where $\alpha \equiv m\omega/\hbar$. With that our expectation values become

$$\langle\Psi|\hat{x}|\Psi\rangle = \frac{1}{\sqrt{2\alpha}}\cos\omega t\,, \qquad \langle\Psi|\hat{x}^2|\Psi\rangle = \frac{1}{\alpha}\,,$$

$$\langle\Psi|\hat{p}|\Psi\rangle = -\hbar\sqrt{\frac{\alpha}{2}}\,\sin\omega t\,, \qquad \langle\Psi|\hat{p}^2|\Psi\rangle = \hbar^2\alpha\,.$$

The product $\Delta x\Delta p$ is then

$$\Delta x\Delta p = \sqrt{\frac{1}{\alpha} - \frac{1}{2\alpha}\cos^2\omega t}\,\sqrt{\hbar^2\alpha - \hbar^2\frac{\alpha}{2}\sin^2\omega t} = \frac{\hbar}{2}\sqrt{(2-\cos^2\omega t)(2-\sin^2\omega t)} \geq \frac{\hbar}{2}\,.$$

Thus, the Heisenberg uncertainty principle does hold for any value of time t.

Problem 2. Some physical system has the observables that are represented by the following operators:

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

(a) What are the possible results of the measurements of these observables?

(b) Which of these observables are mutually compatible? Find a basis of common eigenstates.

(c) Find the operator that, when acting on an arbitrary state, leaves only the component corresponding to the positive values of observable D.

Solution:

(a) The possible results of measuring observables A, B, C, and D are the eigenvalues of the corresponding matrices. Let us find the eigenvalues and the corresponding eigenvectors for all four observables:

$$a_1 = -1\,, \quad |a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad a_2 = 3\,, \quad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad a_3 = 5\,, \quad |a_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$b_1 = -3\,, \quad |b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad b_2 = 1\,, \quad |b_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad b_3 = 3\,, \quad |b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

$$c_1 = -\sqrt{13}\,, \ |c_1\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix} 3 \\ -\sqrt{13} \\ 2 \end{pmatrix}, \quad c_2 = 0\,, \ |c_2\rangle = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad c_3 = \sqrt{13}\,, \ |c_3\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix} 3 \\ \sqrt{13} \\ 2 \end{pmatrix},$$

$$d_1 = -1\,, \quad |d_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \qquad d_2 = 1\,, \quad |d_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad d_3 = 1\,, \quad |d_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$$

Thus, the possible results are:

A: -1, 3, 5

B: -3, 1, 3

C: $-\frac{1}{\sqrt{2}}$, 0, $\frac{1}{\sqrt{2}}$

D: -1, 1, 1.

(b) By direct calculation we can determine that $[A, B] = 0$, i.e. operators A and B commute. This is the only pair of operators that commute in the set of four. Thus only A and B are mutually compatible. Because A and B are compatible, we can choose a common set of eigenstates for them, i.e. these eigenstates are eigenstates of A and B simultaneously:

$$|a_1, b_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad |a_2, b_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad |a_3, b_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Here we essentially need to find a projection operator P that projects on the subspace spanned by $|d_2\rangle$ and $|d_3\rangle$, because these two eigenvectors correspond to positive eigenvalues:

$$P = |d_2\rangle\langle d_2| + |d_3\rangle\langle d_3| = 1 - |d_1\rangle\langle d_1|\,,$$

$$P \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & -i\end{pmatrix} \;+\; \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\end{pmatrix} \;=\; \frac{1}{2}\begin{pmatrix}2 & 0 & 0\\0 & 1 & -i\\0 & i & 1\end{pmatrix}.$$

Problem 3.

(a) Find the eigenstates of the position operator, $\hat{x}$, in the coordinate representation.

(b) Now consider two particles in 1D that are stuck to each other and move with a definite value of the total linear momentum P $(= K\hbar)$. Write the wave function of this system in the coordinate respresentation. You can ignore the normalization.

(c) What is the wave function of the system in the momentum representation?

(d) Suppose we measure the coordinate of the first particle and obtain a. In what state will the second particle be left (i.e. what state will the second particle be projected onto)?

(e) Suppose we measure the momentum of the first particle and obtain p_0 $(= k_0\hbar)$. In what state will the second particle be left?

Solution:

(a) Here we need to solve the equation

$$\hat{A} f_\alpha(x) = \alpha f_\alpha(x),$$

where $\hat{A}$ is the operator, α is its eigenvalue (labelled by α; obviously α can take any real value as this is the position), and $f_\alpha(x)$ is the eigenfunction corresponding to eigenvalue α. In our case $\hat{A} = x$ and the equation becomes

$$(x - \alpha) f_\alpha(x) = 0.$$

By looking at this equation we can see that the solution should be zero at any point $x \neq \alpha$. On the other hand, at point $x = \alpha$ the solution should not be zero (otherwise the solution would be zero everywhere and that does not make sense). Moreover, the wave function should be normalized. The only function that has these properties is the Dirac delta function:

$$f_\alpha(x) = \delta(x - \alpha),$$

(b) Because the particles are stuck to each other the wave function that describes their relative motion is the Dirac delta function $\delta(x_1 - x_2)$. On the other hand, their motion as a whole in 1D, which is completely independent on their relative motion (essentially there is no relative motion – the particles are stuck to each other), is characterized by a definite linear momentum K. We know that the eigenstates of the linear momentum operator are plane waves, e^{iKx}. In our case x can be either x_1 or x_2 – it does not matter because when particles are stuck to each other $x_1 = x_2$. Therefore, the total wave function of the system in the coordinate representation is

$$\langle x_1, x_2 | \psi \rangle \equiv \psi(x_1, x_2) = A\, e^{iKx_1} \delta(x_1 - x_2).$$

This wave function cannot be normalized in the usual sense of the term (just like a plane wave corresponding to a free motion cannot be normalized). However, it can be normalized in the sense that we could say that there is a certain number of particles in a unit volume V, in which case $A = 1/\sqrt{V}$.

(c) To obtain the wave function in the momentum representation we need to do a Fourier transform (in both coordinates, x_1 and x_2):

$$\begin{aligned}
\langle k_1, k_2|\psi\rangle \equiv \tilde{\psi}(k_1, k_2) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \psi(x_1, x_2) e^{-ik_1x_1} e^{-ik_2x_2} dx_1 dx_2 \\
&= \frac{A}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{iKx_1}\delta(x_1 - x_2) e^{-ik_1x_1} e^{-ik_2x_2} dx_1 dx_2 \\
&= \frac{A}{2\pi}\int\limits_{-\infty}^{\infty} e^{i(K-k_1-k_2)x_2} dx_2 \\
&= A\,\delta(k_1 + k_2 - K)\,.
\end{aligned}$$

In the above expression we used the identity $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} dk$.

(d) Measuring the position of the first particle (important: measuring the position of the first particle only, not both) and obtaining a results in the projection of the total wavefunction onto the eigenstate $|x_1\!=\!a\rangle = \delta(x_1 - a)$. This yields

$$\langle x_1\!=\!a|\psi\rangle = \int\limits_{-\infty}^{\infty} \delta(x_1 - a) A\, e^{iKx_1}\delta(x_1 - x_2) dx_1 = Ae^{iKa}\delta(x_2 - a) = Ae^{iKa}|x_2\!=\!a\rangle\,.$$

Thus, the second particle is in the state (apart from a multiplicative constant) correspondng to position a. This makes sense as the particles were stuck to each other.

(e) Measuring the momentum of the first particle results in the projection of the total wavefunction onto the eigenstate $|k_1\!=\!k_0\rangle = \delta(k_1 - k_0)$, where we have defined $k_0 = p_0/\hbar$. Note that $\delta(k_1 - k_0)$ is a wave function in the momentum representation. This yields

$$\langle k_1\!=\!k_0|\psi\rangle = \int\limits_{-\infty}^{\infty} \delta(k_1 - k_0) A\,\delta(k_1 + k_2 - K) dk_1 = A\delta(k_2 + k_0 - K) = A|k_2\!=\!K\!-\!k_0\rangle\,.$$

Thus, the second particle is in the state (apart from a multiplicative constant) corresponding to momentum $K - k_0$ (or just $-k_0$ if measured in the reference frame where the system as a whole is not moving). This also makes sense because the total linear momentum of the system must be K. Indeed, $K + k_0 - k_0 = K$.

Problem 4. Consider two spin 1/2 particles. Suppose the operator that describes the interation between them has the following form

$$V = a + \beta\,\mathbf{S}_1\!\cdot\!\mathbf{S}_2 = a + b\,\boldsymbol{\sigma}_1\!\cdot\!\boldsymbol{\sigma}_2,$$

where a, and β are real constants, $b = \beta\hbar^2/4$, and σ's are Pauli spin matrices. The total spin of the system is $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$.

(a) Can V, $\mathbf{S}^2$, and S_z be measured simultaneously?

(b) Determine the matrix form of V in the uncoupled representation.

(c) Determine the matrix form of V in the coupled representation.

Make sure to indicate clearly in which order you place basis states as the matrix form of V depends on this order.

Solution:

(a) Let us first rewrite V in a different form using the relation $\mathbf{S}^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 - 2\,\mathbf{S}_1\!\cdot\!\mathbf{S}_2$:

$$V = a + \frac{\beta}{2}\big(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\big).$$

V, $\mathbf{S}^2$, and S_z can be measured simultaneously only if the operators commute with each other. We know that $\mathbf{S}^2$ commutes with S_z (this takes place for any angular momentum). We also know that $\mathbf{S}^2$ and S_z commute with $a\hat{1}$ (the first term in V). Therefore, we are left to verify whether $\mathbf{S}^2$ and S_z commute with $\mathbf{S}_1\!\cdot\!\mathbf{S}_2 \propto \mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2$. Apparently they do, because

$$[\mathbf{S}^2, \mathbf{S}^2] = 0\,, \qquad [\mathbf{S}^2, \mathbf{S}_1^2] = 0\,, \qquad [\mathbf{S}^2, \mathbf{S}_2^2] = 0,$$

and

$$[S_z, \mathbf{S}^2] = 0\,, \qquad [S_z, \mathbf{S}_1^2] = [S_{1z} + S_{2z}, \mathbf{S}_1^2] = 0\,, \qquad [S_z, \mathbf{S}_2^2] = [S_{1z} + S_{2z}, \mathbf{S}_1^2] = 0\,.$$

Therefore, we can give a positive answer: yes, V, $\mathbf{S}^2$, and S_z can be measured simultaneously.

(b) The uncoupled basis states are products $\big|s_1 = \tfrac{1}{2}, m_1\big\rangle\,\big|s_2 = \tfrac{1}{2}, m_2\big\rangle$. Let us drop the obvious constant labels s_1 and s_2 and denote $\big|m_i = \pm\tfrac{1}{2}\big\rangle$ as $|\pm\rangle$. Then our four basis states $|m_1, m_2\rangle$ are

$$|+\rangle|+\rangle\,,\ |+\rangle|-\rangle\,,\ |-\rangle|+\rangle\,,\ |-\rangle|-\rangle\,,$$

or simply

$$|+\,+\rangle\,,\ |+\,-\rangle\,,\ |-\,+\rangle\,,\ |-\,-\rangle\,.$$

To evaluate the second term in V, let us express $\mathbf{S}_1\cdot\mathbf{S}_2$ through the lowering and raising operators:

$$S_{ix} = \frac{1}{2}\big(S_{i+} + S_{i-}\big)\,, \qquad S_{iy} = \frac{1}{2i}\big(S_{i+} - S_{i-}\big)\,, \qquad i = 1, 2.$$

$$\begin{aligned}
\mathbf{S}_1\!\cdot\!\mathbf{S}_2 &= S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \\
&= \frac{1}{4}\big(S_{1+} + S_{1-}\big)\big(S_{2+} + S_{2-}\big) - \frac{1}{4}\big(S_{1+} - S_{1-}\big)\big(S_{2+} - S_{2-}\big) + S_{1z}S_{2z} \\
&= \frac{1}{2}\big(S_{1+}S_{2-} + S_{1-}S_{2+}\big) + S_{1z}S_{2z} \\
&= \frac{\hbar^2}{8}\big(\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+} + 2\sigma_{1z}\sigma_{2z}\big)\,.
\end{aligned}$$

Then for V we have

$$V = a + \frac{b}{2}\big(\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+} + 2\sigma_{1z}\sigma_{2z}\big)$$

The general formula that describes the action of the lowering and raising operators on states $|s_i, m_i\rangle$ can be taken from the textbook or lecture notes and is

$$S_{i\pm}|s_i, m_i\rangle = \hbar\sqrt{s_i(s_i+1) - m_i(m_i \pm 1)}\,|s_i, m_i \pm 1\rangle\,.$$

In our case of spin 1/2 particles it takes a particularly simple form:

$$S_{i+}\Big|+\frac{1}{2}\Big\rangle = 0\,,\quad S_{i+}\Big|-\frac{1}{2}\Big\rangle = \hbar\Big|+\frac{1}{2}\Big\rangle\,,\quad S_{i-}\Big|+\frac{1}{2}\Big\rangle = \hbar\Big|-\frac{1}{2}\Big\rangle\,,\quad S_{i-}\Big|-\frac{1}{2}\Big\rangle = 0\,,\qquad i = 1, 2,$$

of, in terms of σ's

$$\sigma_{i+}|+\rangle = 0\,,\quad \sigma_{i+}|-\rangle = 2|+\rangle\,,\quad \sigma_{i-}|+\rangle = 2|-\rangle\,,\quad \sigma_{i-}|-\rangle = 0\,,\qquad i = 1, 2.$$

The action of S_{iz} (or σ_{iz}) is also known and trivial:

$$S_{iz}\Big|\pm\frac{1}{2}\Big\rangle = \pm\frac{\hbar}{2}\Big|\pm\frac{1}{2}\Big\rangle\,,\qquad \sigma_{iz}|\pm\rangle = \pm|\pm\rangle\,,\qquad i = 1, 2.$$

With that the action of V on $|m_1, m_2\rangle$ is

$$V|+\,+\rangle = \big(a+b\big)|+\,+\rangle\,,\qquad V|+\,-\rangle = (a-b)|+\,-\rangle + 2b|-\,+\rangle\,,$$

$$V|-\,+\rangle = (a-b)|-\,+\rangle + 2b|+\,-\rangle\,,\qquad V|-\,-\rangle = \big(a+b\big)|-\,-\rangle\,.$$

Multiplying these by $\langle m_1, m_2|$ on the left we can compute the elements of matrix V:

$$\begin{array}{c c}
 & \begin{array}{cccc} |+\,+\rangle & |+\,-\rangle & |-\,+\rangle & |-\,-\rangle \end{array} \\
\begin{array}{c} \langle +\,+| \\ \langle +\,-| \\ \langle -\,+| \\ \langle -\,-| \end{array} &
\begin{pmatrix} a+b & 0 & 0 & 0 \\ 0 & a-b & 2b & 0 \\ 0 & 2b & a-b & 0 \\ 0 & 0 & 0 & a+b \end{pmatrix}
\end{array}$$

(c) The coupled basis states are $|s, m, s_1, s_2\rangle$, where s denotes the total spin quantum number (possible values are $s = 0, 1$) and m is the quantum number of the projection of the total spin on the z-xis (ranges from $-s$ to s). For brevity we can also drop the obvious constant labels s_1 and s_2. The four basis states are

$$|0\quad 0\rangle\,,\qquad |1\,+1\rangle\,,\qquad |1\quad 0\rangle\,,\qquad |1\,-1\rangle\,.$$

Here it is convenient to adopt the form of V that involves $\mathbf{S}^2$:

$$V = a + \frac{\beta}{2}\big(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\big) = a + \frac{b}{2}(\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}_1^2 - \boldsymbol{\sigma}_2^2)\,.$$

The action of $\mathbf{S}^2$ or $\boldsymbol{\sigma}^2$ on the basis states is

$$\mathbf{S}^2|s, m\rangle = \hbar^2 s(s+1)|s, m\rangle \qquad \boldsymbol{\sigma}^2|s, m\rangle = 4s(s+1)|s, m\rangle\,.$$

$$\mathbf{S}_i^2|s, m\rangle = \frac{3}{4}\hbar^2|s, m\rangle \qquad \boldsymbol{\sigma}_i^2|s, m\rangle = 3|s, m\rangle\,,\qquad i = 1, 2\,.$$

The action of V on the basis states is then

$$V|s,m\rangle = \Big[a + b\big(2s(s+1) - 3\big)\Big]|s,m\rangle\,.$$

With this we can easily determine the matrix form of V:

$$
\begin{array}{cc}
 & \begin{array}{cccc} |0\;\;\,0\rangle & |1\,{+1}\rangle & |1\,{-1}\rangle & |1\,{-1}\rangle \end{array} \\
\begin{array}{c} \langle 0\;\;\,0| \\ \langle 1\,{+1}| \\ \langle 1\;\;\,0| \\ \langle 1\,{-1}| \end{array} & \begin{pmatrix} a-3b & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a+b & 0 \\ 0 & 0 & 0 & a+b \end{pmatrix}
\end{array}
$$