PHYS 451 Quantum Mechanics I (Spring 2020)
Homework #3, due Thursday February 13 in class

1D motion, Quantum harmonic oscillator, Fourier transform

1. Prove that for any 1D potential $V(x)$ that satisfies the following conditions
$$V(x) \to 0 \quad \text{for} \quad x \to \pm\infty,$$
and
$$\int_{-\infty}^{\infty} V(x)dx < 0,$$
there exist at least one bound state with negative energy.
Hint: Use a wave function in the form $\varphi(x) \sim e^{-\alpha|x|}$ with $\alpha > 0$. This wave function can be expanded in terms of the eigenfunctions of the Hamiltonian. You can show that for small enough α the expectation value of the Hamiltonian with function $\varphi(x)$ is negative. Since the average energy for any state is always equal or greater than the ground state energy (show it, too) it proves the existence of the ground state with a negative energy.

2. A particle of mass m moves in the potential
$$V(x) = V_1(x) + V_2(x),$$
where $V_1(x) = \beta x$ and $V_2(x) = \gamma x^2$ (β and γ are some positive constants).
   (a) Find the lowest energy state of this potential, $V(x)$.
   (b) If the particle starts out in the ground state of $V_2(x)$ only (i.e. when $V_1(x)$ is switched off), what is the probability that it will end up in the new ground state of $V(x)$ when $V_1(x)$ is suddenly switched on?

3. Consider a particle of mass m sitting in the ground state of the harmonic oscillator potential. What is the probability of finding the particle in the classically forbidden region? Will the corresponding probability in state $n = 100$ (here n is the quantum number) of the harmonic oscillator be larger or smaller? Why?

4. Find the eigenvalues and eigenfunctions of the Hamiltonian with the potential
$$V(x) = \begin{cases} \frac{m\omega x^2}{2}, & x < 0 \\ \infty, & x \geq 0 \end{cases}$$
Hint: Not much math is actually needed here. Some cleaver arguments can help you solve the problem relatively easily.

5. Consider the wave function that has the following form in the coordinate space,
$$\psi(x) = Ae^{-\beta(x-a)^2},$$
where β and a are some real positive constants.
   (a) Find the normalization factor A.
   (b) Compute the Fourier transfom of this function, $\tilde{\psi}(k)$. Please do the math by yourself in this part (i.e. do not refer to any tables of integrals). $\tilde{\psi}(k)$ is the wave function in the momentum space. Show that it comes out normalized in the k-space.
   (c) Compute the expectation values $\langle p \rangle$ and $\langle p^2 \rangle$ the usual way, i.e. using the wave function in the coordinate space.
   (d) Keeping in mind the de Broglie relation ($p = \hbar k$) repeat the same procedure in the momentum space, i.e. use $\tilde{\psi}(k)$.