PHYS 451 Quantum Mechanics I (Spring 2020)
Homework #4, due Thursday February 20 in class

Dirac delta function, Probability current, Transmission and reflection coefficients

1. Consider the function
$$g(x) = \frac{a}{\pi}\,\frac{\sin^2(x/a)}{x^2}.$$
Show that it can be a valid representation of the Dirac delta function in the limit $a \to 0^+$. Specifically, show that in this limit the following properties hold true:
   (a) $\int\limits_{-\infty}^{+\infty} g(x)f(x)dx = f(0)$ for any reasonably slow-changing function $f(x)$.
   (b) $g(x) = g(-x)$.
   (c) $xg(x) = 0$.
   (d) $g(cx) = \frac{1}{|c|}g(x)$, where c is a constant.
   (e) $g'(-x) = -g'(x)$.
   (f) $xg'(x) = -g(x)$.

2. Consider the 1D Gaussian wave packet moving in free space
$$\psi(x, t\!=\!0) = \frac{1}{\left(\pi a^2\right)^{1/4}} \exp\left[ibx - \frac{x^2}{2a^2}\right],$$
where a and b are some real constants and $a > 0$.
   (a) Calculate the probability current j_x for every point x at time $t = 0$.
   (b) Calculate the parobability density, $\rho(x,t)$, explicitely for any $t > 0$.
   (c) Use this probability density and verify that the continuity equation holds true at $t = 0$.

3. Find the transmission and reflection coefficients for the potential in the form of a step-function:
$$V(x) = \begin{cases} 0 & , x < 0 \\ V_0 & , x \geq 0 \end{cases},$$
where $V_0 > 0$. Assume that the particle is incident from the left and the energy of the particle is greater than V_0. Examine the limiting cases $E \to V_0$ and $E \to \infty$.