Dirac delta function, Probability current, Transmission and reflection coefficients

1. Consider the function
   \[ g(x) = \frac{a \sin^2(x/a)}{x^2}. \]
   Show that it can be a valid representation of the Dirac delta function in the limit \( a \to 0^+ \).
   Specifically, show that in this limit the following properties hold true:
   (a) \( \int_{-\infty}^{+\infty} g(x)f(x)dx = f(0) \) for any reasonably slow-changing function \( f(x) \).
   (b) \( g(x) = g(-x) \).
   (c) \( xg(x) = 0 \).
   (d) \( g(cx) = \frac{1}{|c|}g(x) \), where \( c \) is a constant.
   (e) \( g'(-x) = -g'(x) \).
   (f) \( xg'(x) = -g(x) \).

2. Consider the 1D Gaussian wave packet moving in free space
   \[ \psi(x,t=0) = \frac{1}{(\pi a^2)^{1/4}} \exp \left[ ibx - \frac{x^2}{2a^2} \right], \]
   where \( a \) and \( b \) are some real constants and \( a > 0 \).
   (a) Calculate the probability current \( j_x \) for every point \( x \) at time \( t = 0 \).
   (b) Calculate the probability density, \( \rho(x,t) \), explicitly for any \( t > 0 \).
   (c) Use this probability density and verify that the continuity equation holds true at \( t = 0 \).

3. Find the transmission and reflection coefficients for the potential in the form of a step-function:
   \[ V(x) = \begin{cases} 
   0 & , x < 0 \\
   V_0 & , x \geq 0 
   \end{cases}, \]
   where \( V_0 > 0 \). Assume that the particle is incident from the left and the energy of the particle is greater than \( V_0 \). Examine the limiting cases \( E \to V_0 \) and \( E \to \infty \).