PHYS 451 Quantum Mechanics I (Spring 2020)
Homework #5, due Friday February 28, by 3:00pm

Commutators, Operators, Formalism of quantum mechanics

1. The orbital angular momentum operator is defined as

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix},$$

where $|...|$ stands for the determinant and $\mathbf{e}_i$ are unit vectors along the x, y, and z axes respectively. Find commutators for the following components of $\hat{\mathbf{L}}$:

   (a) $[\hat{L}_x, \hat{L}_x]$
   (b) $[\hat{L}_x, \hat{L}_y]$
   (c) $[\hat{L}_x^2, \hat{x}]$
   (d) $[\hat{L}_y^2, \hat{x}]$

2. Determine if the following operators are linear, hermitian, nonhermitian, or antihermitian (an operator $\hat{O}$ is called antihermitian if $\hat{O}^\dagger = -\hat{O}$):

   (a) $\frac{d}{dx}$
   (b) $\hat{x}\hat{p}_x$
   (c) $i(\hat{A}\hat{B} - \hat{B}\hat{A})$, if it is known that $\hat{A}^\dagger = \hat{A}$ and $\hat{B}^\dagger = \hat{B}$
   (d) $\hat{G} - \hat{G}^\dagger$ (G is some arbitrary operator)
   (e) Scaling operator: $\hat{S}_\alpha f(x) = \sqrt{\alpha} f(\alpha x) \quad (\alpha > 0)$
   (f) Exponentiation operator, $\hat{E}$: $\quad \hat{E} f(x) = e^{f(x)}$
   (g) Complex conjugation operator, $\hat{C}$: $\quad \hat{C} f(x) = f^*(x)$
   (h) Operator that computes the absolute value, $\hat{N}$: $\quad \hat{N} f(x) = |f(x)|$

3. A function of an operator can be defined by means of the Taylor series,

$$F(\hat{A}) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \hat{A}^n.$$

Using this definition find the explicit (and compact) form of the following operators. Alternatively, you may write how they act on an arbitrary function $f(x)$:

   (a) $\exp(i\gamma\hat{I})$
   (b) $\exp\left(\frac{ia\hat{p}}{\hbar}\right)$
   (c) $\exp\left(bx\frac{d}{dx}\right)$

Here $\hat{I}$ is the inversion operator (i.e. $\hat{I} f(x) = f(-x)$), $\hat{p}$ is the momentum operator. γ, a, and b are some real constants.

4. Prove the following relation, where $\hat{A}$ and $\hat{B}$ are some operators:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

5. Consider two state vectors in the 2D Hilbert space: $|a\rangle$ and $|b\rangle$. Vector $|a\rangle$ makes 30$^{\text{o}}$ angle with the x-axis and is normalized to unity. Similarly, vector $|b\rangle$ makes -30^{o} angle with the x-axis and is normalized to unity. Write the expansion of $|a\rangle$ and $|b\rangle$ in terms of the following basis states:

   (a) $|x\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|y\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

   (b) $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

   (c) $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$, $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

   For each of the bases, find the inner product $\langle a|b\rangle$ and verify the completeness relation, $\sum\limits_k |k\rangle\langle k| = \hat{1}$.

6. Consider a linear operator represented by the following matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{pmatrix}.$$

   Find all its eigenvalues λ_k and eigenvectors $|\phi_k\rangle$. Make sure that all eigenvectors are orthonormal. Then verify the spectral decomposition

$$\hat{A} = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|.$$