1. The orbital angular momentum operator is defined as

\[ \hat{L} = \hat{r} \times \hat{p} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix}, \]

where \(|...|\) stands for the determinant and \(\mathbf{e}_i\) are unit vectors along the \(x\), \(y\), and \(z\) axes respectively. Find commutators for the following components of \(\hat{L}\):

(a) \([\hat{L}_x, \hat{L}_x]\)
(b) \([\hat{L}_x, \hat{L}_y]\)
(c) \([\hat{L}_z^2, \hat{x}]\)
(d) \([\hat{L}_y^2, \hat{x}]\)

2. Determine if the following operators are linear, hermitian, nonhermitian, or antihermitian (an operator \(\hat{O}\) is called antihermitian if \(\hat{O}^\dagger = -\hat{O}\)):

(a) \(\frac{d}{dx}\)
(b) \(\hat{x}\hat{p}_x\)
(c) \(i(\hat{A}\hat{B} - \hat{B}\hat{A})\), if it is known that \(\hat{A}^\dagger = \hat{A}\) and \(\hat{B}^\dagger = \hat{B}\)
(d) \(\hat{G} - \hat{G}^\dagger\) (\(\hat{G}\) is some arbitrary operator)
(e) Scaling operator: \(\hat{S}_\alpha f(x) = \sqrt{\alpha} f(\alpha x)\) \((\alpha > 0)\)
(f) Exponentiation operator, \(\hat{E}\): \(\hat{E} f(x) = e^{f(x)}\)

(g) Complex conjugation operator, \(\hat{C}\): \(\hat{C} f(x) = f^*(x)\)

(h) Operator that computes the absolute value, \(\hat{N}\): \(\hat{N} f(x) = |f(x)|\)

3. A function of an operator can be defined by means of the Taylor series,

\[ F(\hat{A}) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} \hat{A}^n. \]

Using this definition find the explicit (and compact) form of the following operators. Alternatively, you may write how they act on an arbitrary function \(f(x)\):

(a) \(\exp(i\gamma \hat{I})\)
(b) \(\exp(\frac{ia\hat{p}}{\hbar})\)
(c) \(\exp(bx \frac{d}{dx})\)

Here \(\hat{I}\) is the inversion operator (i.e. \(\hat{I} f(x) = f(-x)\)), \(\hat{p}\) is the momentum operator. \(\gamma\), \(a\), and \(b\) are some real constants.
4. Prove the following relation, where $\hat{A}$ and $\hat{B}$ are some operators:

$$e^{\hat{A}\hat{B}e^{-\hat{A}}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \ldots$$

5. Consider two state vectors in the 2D Hilbert space: $|a\rangle$ and $|b\rangle$. Vector $|a\rangle$ makes $30^\circ$ angle with the $x$-axis and is normalized to unity. Similarly, vector $|b\rangle$ makes $-30^\circ$ angle with the $x$-axis and is normalized to unity. Write the expansion of $|a\rangle$ and $|b\rangle$ in terms of the following basis states:

(a) $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$, $|L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

For each of the bases, find the inner product $\langle a | b \rangle$ and verify the completeness relation, $\sum_{k} |k\rangle \langle k| = 1$.

6. Consider a linear operator represented by the following matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{pmatrix}.$$ 

Find all its eigenvalues $\lambda_k$ and eigenvectors $|\phi_k\rangle$. Make sure that all eigenvectors are orthonormal. Then verify the spectral decomposition

$$\hat{A} = \sum_{k} \lambda_k |\phi_k\rangle \langle \phi_k|.$$