PHYS 451 Quantum Mechanics I (Spring 2020)
Homework #7, due Monday Apr 20 at 11:59pm

Spherical harmonics, motion in central potential

1. Consider a state with some definite projection (say $m\hbar$) of the angular momentum on the z-axis. Prove that in this state

$$\langle L_x \rangle = \langle L_y \rangle = 0,$$
$$\langle L_x^2 \rangle = \langle L_y^2 \rangle,$$
$$\langle L_x L_y \rangle = -\langle L_y L_x \rangle = im\hbar^2/2,$$

   where $\langle ... \rangle$ stands for the expectation value in that state. To do this it may be helpful to use the raising/lowering operators, $L_\pm$.

2. Consider a particle with spin angular momentum 1/2. Given some arbitrary axis, defined by a unit vector $\mathbf{n} = (n_x, n_y, n_z) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, find spin eigenstates that correspond to the positive and negative projection on this axis. To do this, use the operator $\mathbf{n} \cdot \mathbf{S} \propto \mathbf{n} \cdot \boldsymbol{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$. Now, suppose a particle is prepared in a state that corresponds to the positive projection of the spin on the x-axis. What will be the result of measuring the projection of its spin on axis $\mathbf{n}$?

3. Consider a particle with spin $s = 3/2$. Determine the matrix representation of operators $\mathbf{S}^2$, S_x, S_y, S_z in the basis of eigenstates of S_z. This exercise is similar to what was done in lecture for the case $l = 1$.

4. Consider the following five states in the coupled representation (here $|l\, m\, l_1\, l_2\rangle$ stands for a state with a definite value of the total angular momentum, l, and its projection on the z-axis, m, formed by two particles that have orbital angular momenta l_1 and l_2):

$$|2\,2\,1\,1\rangle, \quad |2\,1\,1\,1\rangle, \quad |2\,0\,1\,1\rangle, \quad |2\,-1\,1\,1\rangle, \quad |2\,-2\,1\,1\rangle$$

   Express these states in the uncoupled representation, i.e. find the corresponding Clebsch-Gordan coefficients. To do that, start with the state that has the largest (or the lowest) projection and use the ladder operators. This and the normalization condition should generate algebraic equations with respect to the unknown Clebsch-Gordan coefficients.

5. Problem 4.25 in Griffiths.