StudentID: ______________________

PHYS 451 Quantum Mechanics I (Spring 2020)
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Midterm Exam 1

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider a particle of mass m inside an infinite square well (box) of length a $(0 < x < a)$. The particle is in the state

$$\psi(x) = A[4\phi_2(x) + 3i\phi_{1000}(x)],$$

where A is a constant, and ϕ_n's are eigenstates of the box.

(a) What is the average energy of the particle?

(b) What is the most probable position of the particle? You can give an approximate answer.

(c) If the right wall of the box is suddenly moved from point $x = a$ to $x = 3a$, what will be the probability of finding the particle in the ground state of the new box? Give an approximate numerical value of this probability (does not need to be very accurate).

Problem 2. Find the probability of transmission and reflection for a particle of mass m and energy E that encounters a potential barrier in the form $V(x) = \alpha\delta(x)$, where $\delta(\ldots)$ is the Dirac delta function and α is a positive constant.

Problem 3. Consider a particle of mass m moving in the harmonic oscillator potential $V(x) = \frac{m\omega^2 x^2}{2}$. At time $t = \pi/\omega$ the particle's wave function is given by $B(1 + \sqrt{\alpha}x)e^{-\alpha x^2/2}$, where $\alpha = m\omega/\hbar$ and B is a positive constant.

(a) Find $\Psi(x, t)$ and make sure it is properly normalized.

(b) If a measurement of the energy is made, what will be the possible outcomes and with what probability they may occur?

(c) Compute the expectation values of the particle's position and momentum at an arbitrary moment of time t.

Problem 4. The polarization of photons can be described by a complex vector in 2D Hilbert space, i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$, where a and b are complex numbers. In particular, we can distiguish several important cases,

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; |R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \; |L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix},$$

which correspond to the horizontal (0°), vertical (90°), diagonal (+45°, −45°), and circular (right, left) polarization respectively. Now, suppose we prepared a beam of photons, all in state $|\psi\rangle$, and send it through a filter that transmits only the photons that correspond to the horizontal polarization. The measured transmission coefficient for state $|\psi\rangle$ is $\mathcal{P}_x = 1/5$. Then we repeat the experiment with two other filters that transmit only the diagonally and circularly polarized light. The measured transmission coefficients are $\mathcal{P}_+ = 1/2$ and $\mathcal{P}_R = 9/10$ respectively.

(a) Based on the outcomes of the experiments with filters, determine state $|\psi\rangle$. Note that without loss of generality you can assume that either a or b is real (because you can factor out a common phase factor, which is arbitrary here anyway). This may simplify your algebraic manipulations.

(b) What will be the transmission coefficient for state $|\psi\rangle$ if we put along the path of light all these filters together – first the filter that transmits only the horizontally polarized light, then the filter that transmits the diagonally polarized light, and, last, the filter that transmits the circular light with right polarization? Note that an action of a filter can be described by the corresponding projection operator.

Appendix: formula sheet

Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\quad (0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\left(2\alpha x^2 - 1\right)e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{k+1}\,\delta_{n,k+1}\right)$

$$\langle\phi_n|\hat{p}^2|\phi_k\rangle = -\frac{m\hbar\omega}{2}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} - (2k+1)\,\delta_{nk} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2}\right)$$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Dirac delta function

$\int\limits_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$ $\delta(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ikx}dk$ $\delta(-x) = \delta(x)$ $\delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} dk$$

or, in terms of $p = \hbar k$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} f(x) e^{-ipx/\hbar} dx \qquad f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \tilde{f}(p) e^{ipx/\hbar} dp$$

Useful integrals

$$\int_0^\infty x^{2k} e^{-\beta x^2} dx = \sqrt{\pi} \frac{(2k)!}{k!\, 2^{2k+1} \beta^{k+1/2}} \quad (\mathrm{Re}\, \beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^{2k+1} e^{-\beta x^2} dx = \frac{1}{2} \frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\, \beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^k e^{-\gamma x} dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\, \gamma > 0,\ k = 0, 1, 2, ...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2} e^{iqx} dx = \sqrt{\frac{\pi}{\beta}} e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\, \beta > 0)$$

$$\int_0^\pi \sin^{2k} x\, dx = \pi \frac{(2k-1)!!}{2^k\, k!} \quad (k = 0, 1, 2, ...)$$

$$\int_0^\pi \sin^{2k+1} x\, dx = \frac{2^{k+1}\, k!}{(2k+1)!!} \quad (k = 0, 1, 2, ...)$$

Useful trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta \qquad \cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$
$$\sin\alpha \sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \cos\alpha \cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$
$$\sin\alpha \cos\beta = \tfrac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad \cos\alpha \sin\beta = \tfrac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Useful identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2 x = 1 \qquad \coth^2 x - \mathrm{csch}^2 x = 1$$