Problem 1. Consider a particle of mass m inside an infinite square well (box) of length a $(0 < x < a)$. The particle is in the state

$$\psi(x) = A[4\phi_2(x) + 3i\phi_{1000}(x)],$$

where A is a constant, and ϕ_n's are eigenstates of the box.

(a) What is the average energy of the particle?

(b) What is the most probable position of the particle? You can ignore very fine details and give an approximate answer.

(c) If the right wall of the box is suddenly moved from point $x = a$ to $x = 3a$, what will be the probability of finding the particle in the ground state of the new box? Give an approximate numerical value of this probability (does not need to be very accurate).

Solution:

First, let us find the value of the normalization constant, A, keeping in mind that ϕ_n's are orthonormal:

$$1 = \int_0^a \psi^*(x)\psi(x)dx = |A|^2 \int_0^a [4\phi_2(x) - 3i\phi_{1000}(x)][4\phi_2(x) + 3i\phi_{1000}(x)]dx =$$

$$= |A|^2 \int_0^a [16\phi_2^2(x) + 9\phi_{1000}^2(x)]dx = |A|^2[16 + 9].$$

So, up to an arbitrary phase factor $A = 1/5$, i.e.

$$\psi(x) = \frac{4}{5}\phi_2(x) + \frac{3i}{5}\phi_{1000}(x).$$

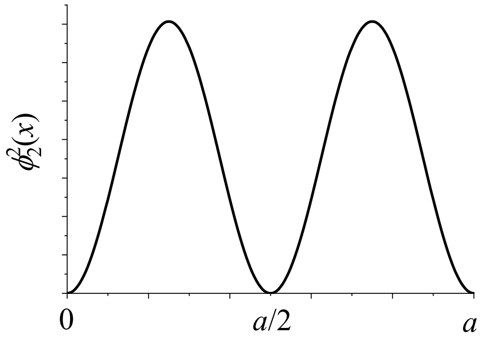
(a) The probability of finding the particle in state $n = 2$ and $n = 1000$ is $\mathcal{P}_2 = |4/5|^2 = 16/25$ and $\mathcal{P}_{1000} = |3i/5|^2 = 9/25$ respectively (the corresponding probabilities for other states are zeros). Therefore, the average energy is

$$\langle E \rangle = \sum_{n=1}^{\infty} \mathcal{P}_n E_n = \mathcal{P}_2 E_2 + \mathcal{P}_{1000} E_{1000} = \frac{16}{25}\frac{2^2\pi^2\hbar^2}{2ma^2} + \frac{9}{25}\frac{1000^2\pi^2\hbar^2}{2ma^2} = \frac{4500032}{25}\frac{\pi^2\hbar^2}{ma^2}.$$

(b) The most probable position corresponds to the maximum of the probability density, $\rho(x)$, which in our case is

$$\rho(x) = \psi^*(x)\psi(x) = \frac{16}{25}\phi_2^2(x) + \frac{9}{25}\phi_{1000}^2(x).$$

The last term in the above expression is highly oscillatory (and periodic). For practical purposes, unless we are looking at extremely fine details, it can be considered constant. Hence, the maxima of $\rho(x)$ will be determined by the maxima of $\phi_2^2(x)$. The latter has two equal maxima at $x = a/4$ and $x = 3a/4$.



(c) The probability of finding the system in the ground state of the new box (of length $3a$) is given by

$$\mathcal{P} = |\langle \phi_1^{(3a)}(x)|\psi(x)\rangle|^2 = \left|\left\langle \phi_1^{(3a)}(x)\middle|\frac{4}{5}\phi_2^{(a)}(x) + \frac{3i}{5}\phi_{1000}^{(a)}(x)\right\rangle\right|^2,$$

where $\phi_n^{(a)}$ and $\phi_n^{(3a)}$ denote the eigenstates of the original box and the new box respectively. The calculations then yield

$$\begin{aligned}\langle \phi_1^{(3a)}(x)|\psi(x)\rangle &= \frac{4}{5}\int_0^a \left(\sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}\right)\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right)dx + \frac{3i}{5}\int_0^a \left(\sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}\right)\left(\sqrt{\frac{2}{a}}\sin\frac{1000\pi x}{a}\right)dx \\ &= \frac{4}{5}\left(-\frac{18}{35\pi}\right) + \frac{3i}{5}\left(-\frac{9000}{8999999\pi}\right).\end{aligned}$$

and

$$\mathcal{P} = \frac{1}{25\pi^2}\left|\frac{72}{35} + \frac{27000i}{8999999}\right|^2 = \frac{1}{25\pi^2}\left[\left(\frac{72}{35}\right)^2 + \left(\frac{27000}{8999999}\right)^2\right] \approx 0.017$$

Problem 2. Find the probability of transmission and reflection for a particle of mass m and energy E that encounters a potential barrier in the form $V(x) = \alpha\delta(x)$, where $\delta(\ldots)$ is the Dirac delta function and α is a positive constant.

Solution:

See pages 5–8 in lecture #8

Problem 3. Consider a particle of mass m moving in the harmonic oscillator potential $V(x) = \frac{m\omega^2 x^2}{2}$. At time $t = \pi/\omega$ the particle's wave function is given by $B(1 + \sqrt{\alpha}x)e^{-\alpha x^2/2}$, where $\alpha = m\omega/\hbar$ and B is a positive constant.

(a) Find $\Psi(x, t)$ and make sure it is properly normalized.

(b) If a measurement of the energy is made, what will be the possible outcomes and with what probability they may occur?

(c) Compute the expectation values of the particle's position and momentum at an arbitrary moment of time t.

Solution:

(a) The wave function (at $t = \pi\omega$) contains only two terms when we represent it as an expansion in terms of the eigenstates of the Hamiltonian, $\phi_n(x)$:

$$\begin{aligned}\Psi(x, t = \pi\omega) &= B(1 + \alpha x)e^{-\alpha x^2/2} = B\frac{\pi^{1/4}}{\alpha^{1/4}}\left(\frac{\alpha^{1/4}}{\pi^{1/4}}e^{-\alpha x^2/2} + \frac{1}{\sqrt{2}}\frac{\alpha^{3/4}}{\pi^{1/4}}\sqrt{2}\,xe^{-\alpha x^2/2}\right) \\ &= B\frac{\pi^{1/4}}{\alpha^{1/4}}\left(\phi_0(x) + \frac{1}{\sqrt{2}}\phi_1(x)\right). \end{aligned} \tag{1}$$

The normalization of the wavefunction obviously requires that $B = \frac{\alpha^{1/4}}{\pi^{1/4}}\sqrt{\frac{2}{3}}$ and we get

$$\Psi(x, t = \pi\omega) = \sqrt{\frac{2}{3}}\,\phi_0(x) + \sqrt{\frac{1}{3}}\,\phi_1(x). \tag{2}$$

Since our potential does not depend on time, we know that the general solution to the time-dependent Schrödinger equation can be written as (up to a common phase factor):

$$\Psi(x, t) = \sum_n C_n\phi_n(x)e^{-iE_nt/\hbar}, \tag{3}$$

where E_n are energy eigenvalues, which for the harmonic oscillator are $E_n = \hbar\omega(n + 1/2)$. Matching expression (2) with expression (3) at $t = \pi/\omega$ gives

$$\sqrt{\frac{2}{3}}\,\phi_0(x) + \sqrt{\frac{1}{3}}\,\phi_1(x) = C_0\phi_0(x)e^{-i\pi/2} + C_1\phi_1(x)e^{-3i\pi/2}.$$

So $C_0 = -i$, $C_1 = i$, and the time-dependent wave function is

$$\Psi(x, t) = -i\sqrt{\frac{2}{3}}\,\phi_0(x)e^{-i\omega t/2} + i\sqrt{\frac{1}{3}}\,\phi_1(x)e^{-3i\omega t/2}.$$

(b) Possible values of energy are $E_0 = \hbar\omega/2$ and $E_1 = 3\hbar\omega/2$, while the corresponding probabilities are

$$\mathcal{P}_0 = \left|-i\sqrt{\frac{2}{3}}\right|^2 = \frac{2}{3},$$

$$\mathcal{P}_1 = \left|i\sqrt{\frac{1}{3}}\right|^2 = \frac{1}{3}.$$

(c) The expectation value of x is

$$\begin{aligned}
\langle x\rangle &= \langle\Psi(x,t)|x|\Psi(x,t)\rangle \\
&= \left\langle \sqrt{\frac{2}{3}}\,\phi_0(x)e^{-i\omega t/2} - \sqrt{\frac{1}{3}}\,\phi_1(x)e^{-3i\omega t/2}\right| x \left|\sqrt{\frac{2}{3}}\,\phi_0(x)e^{-i\omega t/2} - \sqrt{\frac{1}{3}}\,\phi_1(x)e^{-3i\omega t/2}\right\rangle \\
&= \frac{2}{3}\langle\phi_0|x|\phi_0\rangle + \frac{1}{3}\langle\phi_1|x|\phi_1\rangle - \frac{\sqrt{2}}{3}\langle\phi_0|x|\phi_1\rangle e^{-i\omega t} - \frac{\sqrt{2}}{3}\langle\phi_1|x|\phi_0\rangle e^{i\omega t}.
\end{aligned}$$

The diagonal matrix elements of x with eigenfunctions ϕ_n vanish due to symmetry. The off-diagonal matrix elements can be taken from the appendix and they are $\langle\phi_1|x|\phi_0\rangle = \langle\phi_0|x|\phi_1\rangle = \sqrt{\frac{\hbar}{2m\omega}}$. With that we get

$$\langle x\rangle = -\frac{1}{3}\sqrt{\frac{\hbar}{m\omega}}(e^{i\omega t} + e^{-i\omega t}) = -\frac{2}{3}\sqrt{\frac{\hbar}{m\omega}}\,\cos\omega t.$$

The expectation value of $\hat{p} = -i\hbar\frac{d}{dx}$ can be calculated directly (in a similar way as $\langle x\rangle$). However, it is easier to use the relation

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt}$$

instead. It immediately yields

$$\langle p\rangle = \frac{2}{3}\sqrt{\hbar m\omega}\sin\omega t.$$

Problem 4. The polarization of photons can be described by a complex vector in 2D Hilbert space, i.e. $\begin{pmatrix} a \\ b \end{pmatrix}$, where a and b are complex numbers. In particular, we can distiguish several important cases,

$$|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ |R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \ |L\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

which correspond to the horizontal (0^o), vertical (90^o), diagonal ($+45^o$, -45^o), and circular (right, left) polarization respectively. Now, suppose we prepared a beam of photons, all in state $|\psi\rangle$, and send it through a filter that transmits only the photons that correspond to the horizontal polarization. The measured transmission coefficient for state $|\psi\rangle$ is $\mathcal{P}_x = 1/5$. Then we repeat the experiment with two other filters that transmit only the diagonally and circularly polarized light. The measured transmission coefficients are $\mathcal{P}_+ = 1/2$ and $\mathcal{P}_R = 9/10$ respectively.

(a) Based on the outcomes of the experiments with filters, determine state $|\psi\rangle$. Note that without loss of generality you can assume that either a or b is real (because you can factor out a common phase factor, which is arbitrary here anyway). This may simplify your algebraic manipulations.

(b) What will be the transmission coefficient for state $|\psi\rangle$ if we put along the path of light all these filters together – first the filter that transmits only the horizontally polarized light, then the filter that transmits the diagonally polarized light, and, last, the filter that transmits the circular light with right polarization? Note that an action of a filter can be described by the corresponding projection operator.

Solution:

(a) The action of each filter can be decribed by a projection operator Π that extracts from the input, ψ, the component corresponding to a specific polarization, i.e.

$$\Pi_x = |x\rangle\langle x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Pi_+ = |+\rangle\langle +| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$$\Pi_R = |R\rangle\langle R| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

The transmission probabilities are given by the square of the norm of the extracted component (we assume that $\langle\psi|\psi\rangle = 1$):

$$\mathcal{P}_x = \langle \Pi_x\psi|\Pi_x\psi\rangle = \langle\psi|\Pi_x\psi\rangle = \langle\psi|x\rangle\langle x|\psi\rangle = \left|\langle x|\psi\rangle\right|^2,$$

$$\mathcal{P}_+ = \langle \Pi_+\psi|\Pi_+\psi\rangle = \langle\psi|\Pi_+\psi\rangle = \langle\psi|+\rangle\langle +|\psi\rangle = \left|\langle +|\psi\rangle\right|^2,$$

$$\mathcal{P}_R = \langle \Pi_R\psi|\Pi_R\psi\rangle = \langle\psi|\Pi_R\psi\rangle = \langle\psi|R\rangle\langle R|\psi\rangle = \left|\langle R|\psi\rangle\right|^2.$$

In the above formulae we used the fact for a projection operator $\Pi^\dagger = \Pi$ and $\Pi^2 = \Pi$. If we denote $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, where a and b are some complex numbers then

$$\langle x|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a,$$

$$\langle +|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a+b),$$

$$\langle R|\psi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}(a-ib).$$

With that we have three conditions,

$$\mathcal{P}_x = |a|^2 = \frac{1}{5},$$

$$\mathcal{P}_+ = \frac{1}{2}|a+b|^2 = \frac{1}{2},$$

$$\mathcal{P}_R = \frac{1}{2}|a-ib|^2 = \frac{9}{10},$$

supplemented by the normalization condition

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2 = 1.$$

In order to solve a system of four algebraic equations,

$$|a|^2 + |b|^2 = 1, \quad |a|^2 = 1/5, \quad |a+b|^2 = 1, \quad |a-ib|^2 = 9/5,$$

we will assume that $a = \alpha$ is real and positive (we can always achieve that by factoring out and neglecting a common phase factor). On the other hand, we can represent b as $b = \beta + i\gamma$, where β and γ are real numbers. Then the four equations become

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \quad \alpha = 1/\sqrt{5}, \quad (\alpha+\beta)^2 + \gamma^2 = 1, \quad (\alpha+\gamma)^2 + \beta^2 = 9/5$$

and we can easily faind that there is only one combination of β and γ that satisfies the above equations:

$$\beta = 0, \quad \gamma = 2/\sqrt{5}\,.$$

So our state $|\psi\rangle$ is

$$|\psi\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

(b) The transmission probability for the case of all three filters put together is given by

$$\mathcal{P}_{\text{tot}} = \langle \Pi_R\Pi_+\Pi_x\psi|\Pi_R\Pi_+\Pi_x\psi\rangle = \langle\psi|\Pi_x\Pi_+\Pi_R\Pi_R\Pi_+\Pi_x\psi\rangle = \langle\psi|\Pi_x\Pi_+\Pi_R\Pi_+\Pi_x|\psi\rangle,$$

or

$$\mathcal{P}_{\text{tot}} = \langle\psi|x\rangle\langle x|+\rangle\langle +|R\rangle\langle R|+\rangle\langle +|x\rangle\langle x|\psi\rangle = \left|\langle R|+\rangle\langle +|x\rangle\langle x|\psi\rangle\right|^2.$$

By doing all the multiplications we get

$$\langle R|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2}(1-i)\ ,$$

$$\langle +|x\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \ ,$$

$$\langle x|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2i}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}} \ ,$$

and, finally,

$$\mathcal{P}_{\text{tot}} = \left|\frac{1-i}{2\sqrt{10}}\right|^2 = \frac{1}{20}.$$