PHYS 451 Quantum Mechanics I (Spring 2020)
Online Quiz #5

Compute or simplify the following expressions (here α is a real constant and σ's are Pauli matrices):

1. $e^{-i\alpha\hat{\sigma}_x}$
2. $e^{i\alpha\hat{\sigma}_x}\hat{\sigma}_z e^{-i\alpha\hat{\sigma}_x}$

Solution:

1. We can use the Taylor expansion of the exponent and split it into the even and odd powers of $\hat{\sigma}_x$:

$$e^{-i\alpha\hat{\sigma}_x} = \sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\alpha^k\hat{\sigma}_x^k = \sum_{n=0}^{\infty}\frac{(-i)^{2n}}{(2n)!}\alpha^{2n}\hat{\sigma}_x^{2n} + \sum_{n=0}^{\infty}\frac{(-i)^{2n+1}}{(2n+1)!}\alpha^{2n+1}\hat{\sigma}_x^{2n+1}$$

Now $\hat{\sigma}_x^2 = \hat{1}$, $\hat{\sigma}_x^{2n} = \hat{1}$, and $\hat{\sigma}_x^{2n+1} = \hat{\sigma}_x$ (see lecture notes on the properties of the Pauli matrices). This yields

$$\begin{aligned} e^{-i\alpha\hat{\sigma}_x} &= \hat{1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\alpha^{2n} - i\hat{\sigma}_x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\alpha^{2n+1} \\ &= \hat{1}\cos\alpha - i\hat{\sigma}_x\sin\alpha \end{aligned}$$

2. Using the result in part (1) for the exponent, we can write

$$\begin{aligned} e^{i\alpha\hat{\sigma}_x}\hat{\sigma}_z e^{-i\alpha\hat{\sigma}_x} &= (\cos\alpha + i\hat{\sigma}_x\sin\alpha)\,\hat{\sigma}_z\,(\cos\alpha - i\hat{\sigma}_x\sin\alpha) \\ &= \hat{\sigma}_z\cos^2\alpha + \hat{\sigma}_x\hat{\sigma}_z\hat{\sigma}_x\sin^2\alpha + i\,[\hat{\sigma}_x,\hat{\sigma}_z]\sin\alpha\cos\alpha \end{aligned}$$

But $\hat{\sigma}_x\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_x$, and $[\hat{\sigma}_x,\hat{\sigma}_z] = -2i\hat{\sigma}_y$. This allows to simplify it further:

$$\begin{aligned} e^{i\alpha\hat{\sigma}_x}\hat{\sigma}_z e^{-i\alpha\hat{\sigma}_x} &= \hat{\sigma}_z\cos^2\alpha - \hat{\sigma}_z\hat{\sigma}_x^2\sin^2\alpha + 2\hat{\sigma}_y\sin\alpha\cos\alpha \\ &= \hat{\sigma}_z\left(\cos^2\alpha - \sin^2\alpha\right) + \hat{\sigma}_y\sin(2\alpha) \\ &= \hat{\sigma}_z\cos(2\alpha) + \hat{\sigma}_y\sin(2\alpha) \end{aligned}$$