Name: ____________________

PHYS 452 - Quantum Mechanics II (Spring 2015)
Instructor: Sergiy Bubin
Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are attached.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. An attractive potential in 3D does not always have bound states. The existence of bound states may depend on the strength and range of the potential. Consider a particle of mass m moving in the central potential $V(r) = -g\exp(-2br)$, where g and b are positive constants. Now assume that b is fixed and use a simple exponential variational ansatz to estimate the range of g values for which the potential definitely supports bound states.
Information that might be useful: Quartic equation $(x+p)^4 - qx = 0$ with constants $p > 0$ and $q > 0$ has real roots only when $q > \frac{256}{27}p^3$.

Problem 2. Consider a two-dimensional harmonic oscillator with the Hamiltonian

$$H^0 = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2} + \frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2}$$

Now add an anharmonic term to H^0:

$$H' = 2\lambda x^2 y^2$$

(a) Let Π be an operator that exchanges coordinates x and y. Does Π commute with H^0? Does it commute with the total Hamiltonian, $H = H^0 + H'$? What can you deduce about the symmetry properties of the energy eigenfunctions of H^0 and H?

(b) Using the perturbation theory calculate the energy correction for the ground state level to first order in λ

(c) Find the first order correction in λ to the ground state wave function. Make sure you count *all* nonvanishing terms. Is the behavior of the perturbed ground state wave function consistent with your expectation from part (a)?

(d) Using the perturbation theory calculate the energy correction for the first excited energy level to first order in λ

Problem 3. A particle of mass m moves on a ring of length L. It is subject to a potential $V(x)$ $(0 \le x \le L)$, with $V(0) = V(L)$. Here x is the coordinate specifying the position of the particle on the ring. Assume that the particle has a sufficiently large energy, such that $E > V(x)$ for any value of x.

(a) Within the WKB approximation, how many degenerate states are there for each energy eigenvalue? What is (are) the wave function(s) for the state(s) with energy E? (Do not worry about the normalization)

(b) What is the quantization condition on the energy eigenvalue E?

(c) Assume now that the explicit form of $V(x)$ is given by

$$V(x) = \begin{cases} \alpha x, & 0 \le x \le L/2 \\ \alpha(L-x), & L/2 \le x \le L \end{cases}$$

where α is a constant. Find the energy levels for such a potential in the WKB approximation.

Problem 4. Consider a particle of mass m in an infinite square well $(0 \le x \le a)$. At the time $t = 0$ the particle is in the ground state $(n = 1)$. Then at $t > 0$ a weak time-dependent external potential is turned on

$$H'(x,t) = \lambda x e^{-t/\tau},$$

where τ is a constant. To lowest order in λ determine the following transition probabilities at $t \gg \tau$:

(a) $P_{1\to 2}$

(b) $P_{1\to 3}$

(c) $P_{1\to 4}$

Problem 5. The structure of a crystal may be investigated by scattering particles from it. Suppose an incident particle sees the following potential

$$V(\mathbf{r}) = \sum_j v(\mathbf{r} - \mathbf{R}_j),$$

where $\mathbf{R}_j$ are the positions of the scattering atoms and $v(\mathbf{r})$ is the scattering potential of a single atom. Assume that $v(\mathbf{r})$ is weak.

(a) Compute the differential cross section, $\frac{d\sigma_{\text{atom}}}{d\Omega}$, for scattering from a single atom if $v(\mathbf{r}) = ge^{-\alpha r^2}$ (g and α are constants)

(b) Now consider the full potential created by all atoms. What is the differential cross section in terms of $\frac{d\sigma_{\text{atom}}}{d\Omega}$?

Problem 6. Consider an electron (spin-1/2 particle) at rest. Its spin is free to rotate in response to a time-dependent magnetic field. At $t = 0$ there is a magnetic field B_0 in the z-direction and the spin is aligned along the magnetic field, i.e.

$$|\chi(0)\rangle = |\downarrow\rangle.$$

Now suppose that the magnetic field is *very slowly* decreased to zero and then increased in the opposite direction up to the same magnitude. In addition, there is a weak constant ambient field in the x-direction, B', such that $|B'| \ll |B_0|$. The total time-dependent magnetic field can then be written as

$$\mathbf{B}(t) = \begin{cases} (B', 0, B_0), & t < 0 \\ (B', 0, B_0 - \beta t), & 0 \le t \le t_f \\ (B', 0, -B_0), & t > t_f \end{cases},$$

where $t_f = 2B_0/\beta$ and β is some constant.

(a) Write the Hamiltonian of the system for the period of time when the magnetic field is changing.

(b) Use the adiabatic theorem to determine the state of the system at $t > t_f$. Explain whether the result would be the same if there were no ambient field B' present.

(c) What is the condition on the parameters of the problem for the adiabatic theorem to apply?

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1,2,...,\infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\quad (0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\left(2\alpha x^2 - 1\right)e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $\quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $\quad Y_1^{\mp 1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\mp i\phi} = \pm\sqrt{\frac{3}{8\pi}}\,\frac{x\mp iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\quad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ $\quad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$ $\quad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ $\quad$ Action: $\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1 + m_2 = M$$

Pauli matrices

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Electron in a magnetic field

Hamiltonian: $H = \frac{e}{m}\mathbf{B}\cdot\mathbf{S} = \mu_\mathrm{B}\,\mathbf{B}\cdot\boldsymbol{\sigma}$ $\quad \mu_\mathrm{B} = \frac{e\hbar}{2m}$

Stationary perturbation theory formulae

$H = H^0 + \lambda H'$, $\quad E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots$, $\quad \psi_n = \psi_n^{(0)} + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + \ldots$

$$E_n^{(1)} = H'_{nn}$$

$$\psi_n^{(1)} = \sum_m c_{nm}\psi_m^{(0)}, \quad c_{nm} = \begin{cases} \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}}, & n \neq m \\ 0, & n = m \end{cases}$$

$$E_n^{(2)} = \sum_{m\neq n}\frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\psi_n^{(2)} = \sum_m d_{nm}\psi_m^{(0)}, \quad d_{nm} = \begin{cases} \frac{1}{E_n^{(0)} - E_m^{(0)}}\left(\sum_{k\neq n}\frac{H'_{mk}H'_{kn}}{E_n^{(0)} - E_k^{(0)}}\right) - \frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)} - E_m^{(0)}\right)^2}, & n \neq m \\ 0, & n = m \end{cases}$$

WKB wave function

$\psi(x) = \frac{C}{\sqrt{p(x)}}\exp\left(\pm\frac{i}{\hbar}\int p(x)dx\right)$, where $p(x) = \sqrt{2m(E - V(x))}$

Bohr-Sommerfeld quantization rules

$\int_a^b p(x)dx = (n - \frac{1}{2})\pi\hbar$ $\quad$ where a and b are classical turning points and $n = 1, 2, 3, \ldots$

If the potential has vertical walls on one or both sides then the above equation becomes

$\int_a^b p(x)dx = (n - \frac{1}{4})\pi\hbar$ $\quad$ or $\quad \int_a^b p(x)dx = n\pi\hbar$ respectively.

Semiclassical barrier tunneling

$$T \sim \exp\left[-2\int_a^b \kappa(x)dx\right] \qquad \kappa(x) = \frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

Time-dependence of the wave function

$$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda\sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|H'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)}-E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

If $c_n(t_0) = \delta_{nm}$ (e.g. $\psi(\mathbf{r},t_0) = \varphi_m(\mathbf{r})$, where φ_m is an egenfunction of H^0) and $\lambda H'$ is small then at time $t > t_0$
$c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$
where

$$c_n^{(0)} = \delta_{nm}, \qquad c_n^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} H'_{nm}(t')\, e^{i\omega_{nm}t'}dt',$$

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2\sum_k\int_{t_0}^{t}dt'\int_{t_0}^{t'} H'_{nk}(t')H'_{km}(t'')\, e^{i\omega_{nk}t'}e^{i\omega_{km}t''}dt'', \qquad \ldots$$

Fermi's golden rule

Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar}|H'_{fi}|^2\, g(E_f)$, Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar}|H'_{fi}|^2\, g(E_f)$

Adiabatic theorem and geometric phase

Under adiabatic change of a set of parameters $\{R\}$ in a Hamiltonian $H(\{R\})$, if the system is initially (at $t=0$) in the n-th nondegenerate energy eigenstate, it stays in the same energy eigenstate as the parameters change and acquires a phase factor $\psi(t) = e^{-i\theta_n(t)+i\gamma_n(t)}|\psi_n(\{R(t)\})\rangle$, where $\theta_n = \frac{1}{\hbar}\int_0^t E_n(\{R(t')\})dt'$ and $\gamma_n(t) = i\int_0^t\left\langle\psi_n(\{R(t')\})\middle|\frac{\partial}{\partial t'}\psi_n(\{R(t')\})\right\rangle dt'$.
Note: for the exactly degenerate states the transition amplitudes generally do not vanish, no matter how slowly the Hamiltonian is changed.

Stationary quantum scattering

Wave function at $r\to\infty$: $\psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$, $\quad k = \frac{\sqrt{2mE}}{\hbar}$
Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ $\qquad$ Total cross section: $\sigma_{\rm tot} = \int\frac{d\sigma}{d\Omega}d\Omega$

Partial wave analysis

For a spherically symmetric potentials $\psi(r,\theta) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)P_l(\cos\theta)\right]$

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$$

$$\sigma_{\rm tot} = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$
Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'$,
where $\varphi(\mathbf{r})$ is the free-particle solution (incident plane wave)
and $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is the Green's function

Born approximation

$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}', \quad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad q = 2k\sin\frac{\theta}{2}, \quad \mathbf{k} = k\hat{\mathbf{r}}, \quad \mathbf{k}' = k\hat{\mathbf{z}}$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int\limits_0^\infty rV(r)\sin(qr)dr$

Legendre polynomials

$P_0(x) = 1,\;\; P_1(x) = x,\;\; P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\;\; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,\;\; \ldots,\;\; P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$

Orthogonality: $\int\limits_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$

Spherical Bessel, Neumann, and Hankel functions

$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad \ldots, \quad j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$
$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad \ldots, \quad n_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$
$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$
$h_0^{(1)}(x) = -i\frac{e^{ix}}{x}, \quad h_1^{(1)}(x) = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix}, \quad h_2^{(1)}(x) = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix}, \quad \ldots$
$h_0^{(2)}(x) = i\frac{e^{-ix}}{x}, \quad h_1^{(2)}(x) = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}, \quad h_2^{(2)}(x) = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{-ix}, \quad \ldots$
For $x \ll 1$: $j_l(x) \to \frac{2^l l!}{(2l+1)!}x^l, \quad n_l \to -\frac{(2l)!}{2^l l!}x^{-l-1}$
For $x \gg 1$: $h_l^{(1)} \to \frac{1}{x}(-i)^{l+1}e^{ix}, \quad h_l^{(2)} \to \frac{1}{x}(i)^{l+1}e^{-ix}$

Dirac delta function

$\int\limits_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$

Useful integrals

$\int\limits_0^\infty x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\frac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\; k = 0,1,2,...)$

$\int\limits_0^\infty x^{2k+1}e^{-\beta x^2}dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\; k = 0,1,2,...)$

$\int\limits_0^\infty x^{k}e^{-\gamma x}dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\; k = 0,1,2,...)$

$\int\limits_{-\infty}^\infty e^{-\beta x^2}e^{iqx}dx = \sqrt{\frac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$

$\int\limits_0^\pi \sin^{2k}x\,dx = \pi\frac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$

$\int\limits_0^\pi \sin^{2k+1}x\,dx = \frac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$

Useful trigonometric identities

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$