Name: ______________________

PHYS 452: Quantum Mechanics II, Quiz #4

Instruction: use additional sheets if you find it necessary

A particle is incident on a central potential $V(r)$, which is infinitely high at $r \le a$ and vanish at $r > a$, i.e.

$$V(r) = \begin{cases} 0, & r > a \\ \infty, & r \le a \end{cases}$$

Find the total cross section when the energy of the incident particle is low. Define what "low" means in this context.

Appendix:

Stationary quantum scattering

Wave function at $r \to \infty$: $\psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right]$, $\quad k = \frac{\sqrt{2mE}}{\hbar}$

Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ $\qquad$ Total cross section: $\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega$

Partial wave analysis

For a spherically symmetric potentials $\psi(r,\theta) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)P_l(\cos\theta)\right]$

$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$

$\sigma_{\text{tot}} = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'$,
where $\varphi(\mathbf{r})$ is the free-particle solution (incident plane wave)
and $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is the Green's function

Born approximation

$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}'$, $\quad \mathbf{q} = \mathbf{k}' - \mathbf{k}$, $\quad q = 2k\sin\frac{\theta}{2}$, $\quad \mathbf{k} = k\hat{\mathbf{r}}$, $\quad \mathbf{k}' = k\hat{\mathbf{z}}$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} rV(r)\sin(qr)dr$

Legendre polynomials

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$, $\ldots$, $P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$

Orthogonality: $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$

Spherical Bessel, Neumann, and Hankel functions

$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad \ldots, \quad j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$
$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad \ldots, \quad n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$
$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$
$h_0^{(1)}(x) = -i\frac{e^{ix}}{x}, \quad h_1^{(1)}(x) = \left(-\frac{i}{x^2} - \frac{1}{x}\right) e^{ix}, \quad h_2^{(1)}(x) = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{ix}, \quad \ldots$
$h_0^{(2)}(x) = i\frac{e^{-ix}}{x}, \quad h_1^{(2)}(x) = \left(\frac{i}{x^2} - \frac{1}{x}\right) e^{-ix}, \quad h_2^{(2)}(x) = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{-ix}, \quad \ldots$
For $x \ll 1$: $j_l(x) \to \frac{2^l l!}{(2l+1)!} x^l, \quad n_l \to -\frac{(2l)!}{2^l l!} x^{-l-1}$
For $x \gg 1$: $h_l^{(1)} \to \frac{1}{x}(-i)^{l+1} e^{ix}, \quad h_l^{(2)} \to \frac{1}{x}(i)^{l+1} e^{-ix}$

Useful integrals

$$\int_0^\infty x^{2k} e^{-\beta x^2} dx = \sqrt{\pi}\frac{(2k)!}{k!\, 2^{2k+1} \beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$$
$$\int_0^\infty x^{2k+1} e^{-\beta x^2} dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0, 1, 2, ...)$$
$$\int_0^\infty x^k e^{-\gamma x} dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0, 1, 2, ...)$$
$$\int_{-\infty}^\infty e^{-\beta x^2} e^{iqx} dx = \sqrt{\frac{\pi}{\beta}} e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$
$$\int_0^\pi \sin^{2k} x\, dx = \pi\frac{(2k-1)!!}{2^k\, k!} \quad (k = 0, 1, 2, ...)$$
$$\int_0^\pi \sin^{2k+1} x\, dx = \frac{2^{k+1}\, k!}{(2k+1)!!} \quad (k = 0, 1, 2, ...)$$

Useful trigonometric identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$