StudentID: ____________________

PHYS 452: Quantum Mechanics II – Fall 2016
Instructor: Sergiy Bubin
Final Exam

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are provided in the appendix. Please look through this appendix *before* you begin working on the problems.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1.

(a) Compute the best estimate of the ground state energy of a quantum system with the potential $V(x) = \beta|x|$ $(\beta > 0)$ using the trial wave function in the form:

$$\psi(x) = \begin{cases} C\left(1 - \frac{|x|}{a}\right), & -a < x < a \\ 0, & |x| > a \end{cases}$$

(b) Explain (be specific) why a trial wave function in the form

$$\psi(x) = \begin{cases} C, & -a < x < a \\ 0, & |x| > a \end{cases}$$

is not a good choice for estimating the ground state energy

(c) In the spirit of part (a), give an expession for a trial wave function suitable for estimating the energy of the first excited state. There is no need to do any calculations and produce an estimate here, just give an expression for a trial function.

Problem 2. Consider a four-level, partially degenerate system with Hamiltonian H_0. This system is subjected to an additional interaction V. In some basis they look as follows:

$$H_0 = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad V = b \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b \ll a.$$

Find the energy levels of the perturbed system up to the second order in b. Make sure to explain why you choose certian things in your calculations.

Problem 3. A spinless particle of mass m and charge q in a central field is prepared in an s-state ($l = 0$, $m_l = 0$). This state is degenerate in energy with a p-level ($l = 1$, $m_l = 1, 0, -1$). At time $t = 0$ an electric field

$$\boldsymbol{\mathcal{E}}(t) = (0,\, 0,\, \mathcal{E}_0 \sin \omega t)$$

is turned on. Ignoring the existence of states other than the above-mentioned four ones, but making no further approximations, answer the following questions:

(a) What is the Hamiltonian $\hat{H}'$ that describes the interaction of the particle with the electric field?

(b) How does this Hamiltonian look like in the matrix form, in the basis of the above-mentioned four states (you can denote them as $|0\,0\rangle$, $|1\,1\rangle$, $|1\,0\rangle$, $|1\,{-1}\rangle$, or simply as $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$)? Identify clearly which matrix elements vanish based on the symmetry of the integrand. Express the non-vanishing ones in terms of $\langle i|z|j\rangle$.

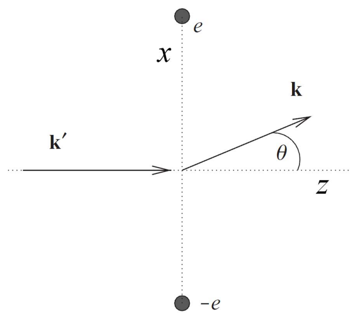
(c) What are the occupation probabilities for each of the four states at time t?

Problem 4. A particle of mass m is free to move in one dimension in the region $-a < x < a$, but it experiences harmonic forces beyond this range. The corresponding potential can be written in the following form:

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2(x+a)^2, & x < -a \\ 0, & -a < x < a \\ \frac{1}{2}m\omega^2(x-a)^2, & x \geq a \end{cases}$$

Find the approximate energy levels of the particle using the semiclassical approach. Take the limits of your result for the case of very small and very large a and show that you get exactly what is expected.

Problem 5. Consider an electric dipole consisting of two opposite charges e and $-e$ fixed at positions $\mathbf{a}$ and $-\mathbf{a}$ from the origin (so that the separation between the two charges is $2a$). A particle of mass m and charge e with an incident wave vector $\mathbf{k}'$ that is perpendicular to the direction of the dipole is scattered off this target. Find the scattering amplitude in the case when the incident energy is very high. What are the angle(s) in xz-plane for which the differential cross section is maximal?



Problem 6. The Schrödinger equation for a particle of mass m moving in 1D in the attractive delta function potential

$$V(x) = -\alpha\delta(x) \qquad \alpha > 0,$$

has a single bound state solution, $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}e^{-\frac{m\alpha}{\hbar^2}|x|}$.

(a) What is the bound state energy?

(b) Compute the geometric phase change when α is decreased very slowly from α_i to α_f.

(c) Compute the dynamic phase change for the same process if α is changed at a constant rate, i.e. $\frac{d\alpha}{dt} = \beta$.

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ \ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \quad (0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x) x\,\phi_k(x) dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\left(2\alpha x^2 - 1\right)e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} - \sqrt{n}\,\delta_{k,n-1}\right)$

Creation and annihilation operators for harmonic oscillator

$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ $[\hat{a},\hat{a}^\dagger] = 1$

$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}$ $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R_{nl}}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions R_{nl} for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{mZe^2}$)

$R_{10} = 2a^{-3/2}\, e^{-\frac{r}{a}} \qquad R_{20} = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a}\right) e^{-\frac{r}{2a}} \qquad R_{21} = \frac{1}{\sqrt{24}} a^{-3/2}\, \frac{r}{a}\, e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r} \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\, \frac{x \pm iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{\mathbf{L}}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ Action: $\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1) - m(m \pm 1)}\,|j, m \pm 1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1 + m_2 = M$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Electron in a magnetic field

Hamiltonian: $H = -\boldsymbol{\mu}\cdot\mathbf{B} = -\gamma\,\mathbf{B}\cdot\mathbf{S} = \frac{e}{m}\mathbf{B}\cdot\mathbf{S} = \mu_{\mathrm{B}}\,\mathbf{B}\cdot\boldsymbol{\sigma}$
here $e > 0$ is the magnitude of the electron electric charge and $\mu_{\mathrm{B}} = \frac{e\hbar}{2m}$

Bloch theorem for periodic potentials $V(x+a) = V(x)$

$\psi(x+a) = e^{iKa}\psi(x)$

Rayleigh-Ritz variational method

$$\psi_{\mathrm{trial}} = \sum_{i=1}^{n} c_i\phi_i \qquad Hc = \epsilon Sc, \quad \text{where } c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad \begin{array}{l} H_{ij} = \langle\phi_i|\hat{H}|\phi_j\rangle \\ S_{ij} = \langle\phi_i|\phi_j\rangle \end{array}$$

Stationary perturbation theory formulae

$\hat{H} = \hat{H}^0 + \lambda\hat{H}', \qquad E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots, \qquad \psi_n = \psi_n^{(0)} + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + \ldots$

$$E_n^{(1)} = H'_{nn}$$

$$\psi_n^{(1)} = \sum_m c_{nm}\psi_m^{(0)}, \quad c_{nm} = \begin{cases} \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}}, & n \neq m \\ 0, & n = m \end{cases}$$

$$E_n^{(2)} = \sum_{m\neq n} \frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\psi_n^{(2)} = \sum_m d_{nm}\psi_m^{(0)}, \quad d_{nm} = \begin{cases} \frac{1}{E_n^{(0)} - E_m^{(0)}}\left(\sum_{k\neq n} \frac{H'_{mk}H'_{kn}}{E_n^{(0)} - E_k^{(0)}}\right) - \frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)} - E_m^{(0)}\right)^2}, & n \neq m \\ 0, & n = m \end{cases}$$

WKB wave function

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \exp\left[+\frac{i}{\hbar}\int p(x)dx\right] + \frac{B}{\sqrt{p(x)}} \exp\left[-\frac{i}{\hbar}\int p(x)dx\right], \text{ where } p(x) = \sqrt{2m(E-V(x))}$$

Bohr-Sommerfeld quantization rules

$\int_a^b p(x)dx = (n-\frac{1}{2})\pi\hbar$ – the potential has no vertical walls at a or b

$\int_a^b p(x)dx = (n-\frac{1}{4})\pi\hbar$ – only one wall of the potential is vertical

$\int_a^b p(x)dx = n\pi\hbar$ – both walls of the potential are vertical

Here a and b are classical turning points and $n = 1, 2, 3, \ldots$

Semiclassical barrier tunneling

$$T \sim \exp\left[-2\int_a^b \kappa(x)dx\right] \qquad \kappa(x) = \frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

General time-dependence of the wave function (TDSE in matrix form)

$$\hat{H}(\mathbf{r},t) = \hat{H}^0(\mathbf{r}) + \lambda\hat{H}'(\mathbf{r},t), \qquad \hat{H}^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda\sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|\hat{H}'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)}-E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

$\hat{H}(\mathbf{r},t) = \hat{H}^0(\mathbf{r}) + \lambda\hat{H}'(\mathbf{r},t), \qquad \hat{H}^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \lambda\hat{H}'$ is small

$\psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}}, \qquad c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$

If $c_n(t_0) = \delta_{nm}$ then at time $t > t_0$

$$c_n^{(0)} = \delta_{nm},$$

$$c_n^{(1)}(t) = \frac{1}{i\hbar}\int_{t_0}^t H'_{nm}(t')\,e^{i\omega_{nm}t'}dt',$$

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2\sum_k\int_{t_0}^t dt'\int_{t_0}^{t'} H'_{nk}(t')H'_{km}(t'')\,e^{i\omega_{nk}t'}e^{i\omega_{km}t''}dt'', \qquad \ldots$$

Fermi's golden rule

Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar}|\mathcal{H}'_{fi}|^2\,g(E_f)$, Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\mathcal{H}'_{fi}|^2\,g(E_f)$
where $\mathcal{H}'_{fi} = \langle\varphi_f|\hat{\mathcal{H}}'(\mathbf{r})|\varphi_i\rangle$ and $g(E)$ is the density of states

Stationary quantum scattering

Wave function at $r\to\infty$: $\psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right], \quad k = \frac{\sqrt{2mE}}{\hbar}$
Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ Total cross section: $\sigma_{\text{tot}} = \int\frac{d\sigma}{d\Omega}d\Omega$

Partial wave analysis

For a spherically symmetric potential $\psi(r,\theta) = A\left[e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr)P_l(\cos\theta)\right]$

$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$

$\sigma_{\text{tot}} = 4\pi\sum_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'$,

where $\varphi(\mathbf{r})$ – free-particle solution (incident wave), $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ – Green's function

Born approximation

$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}', \quad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad q = 2k\sin\frac{\theta}{2}, \quad \mathbf{k} = k\hat{\mathbf{r}}, \quad \mathbf{k}' = k\hat{\mathbf{z}}$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^{\infty} rV(r)\sin(qr)dr$

Adiabatic evolution of a particle that starts in the k-th state of a time-dependent Hamiltonian $\hat{H}(t)$

$\Psi_k(\mathbf{r},t) = e^{i\theta_k(t)}e^{i\gamma_k(t)}\psi_k(\mathbf{r},t), \quad \hat{H}(t)\psi_k(\mathbf{r},t) = E_k(t)\psi_k(\mathbf{r},t), \quad \theta_k(t) = -\frac{1}{\hbar}\int_0^t E_k(t')dt'$,

$\gamma_k(t) = i\int_0^t \langle\psi_k(\mathbf{r},t')|\frac{\partial}{\partial t'}\psi_k(\mathbf{r},t')\rangle dt' = i\int_{\mathbf{R}(0)}^{\mathbf{R}(t)}\langle\psi_k|\nabla_{\mathbf{R}}\psi_k\rangle\cdot d\mathbf{R}, \quad \mathbf{R}(t) = (R_1(t), R_2(t), \ldots, R_N(t))$,

$R_i(t),\ i = 1,\ldots,N$ are parameters in the Hamiltinian that change with time

Dirac delta function

$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \frac{1}{|c|}\delta(x)$

Fourier transform conventions

$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{f}(k)e^{ikx}dk$

or, in terms of $p = \hbar k$

$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} f(x)e^{-ipx/\hbar}dx \qquad f(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty}\tilde{f}(p)e^{ipx/\hbar}dp$

Legendre polynomials

$P_0(x) = 1,\ \ P_1(x) = x,\ \ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\ \ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,\ \ \ldots,\ \ P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$

Orthogonality: $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$

Spherical Bessel equation

$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{dR(r)}{dr} + [k^2r^2 + l(l+1)]R(r) = 0$

Spherical Bessel, Neumann, and Hankel functions

$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad \ldots, \quad j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$
$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad \ldots, \quad n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$
$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$
$h_0^{(1)}(x) = -i\frac{e^{ix}}{x}, \quad h_1^{(1)}(x) = \left(-\frac{i}{x^2} - \frac{1}{x}\right) e^{ix}, \quad h_2^{(1)}(x) = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{ix}, \quad \ldots$
$h_0^{(2)}(x) = i\frac{e^{-ix}}{x}, \quad h_1^{(2)}(x) = \left(\frac{i}{x^2} - \frac{1}{x}\right) e^{-ix}, \quad h_2^{(2)}(x) = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right) e^{-ix}, \quad \ldots$
For $x \ll 1$: $j_l(x) \to \frac{2^l l!}{(2l+1)!} x^l, \quad n_l \to -\frac{(2l)!}{2^l l!} x^{-l-1}$
For $x \gg 1$: $h_l^{(1)} \to \frac{1}{x}(-i)^{l+1} e^{ix}, \quad h_l^{(2)} \to \frac{1}{x}(i)^{l+1} e^{-ix}$

Useful integrals

$$\int x \sin(\alpha x)\, dx = \frac{\sin(\alpha x)}{\alpha^2} - \frac{x\cos(\alpha x)}{\alpha}$$

$$\int x^2 \sin(\alpha x)\, dx = \frac{2x\sin(\alpha x)}{\alpha^2} - \frac{\left(\alpha^2 x^2 - 2\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^3 \sin(\alpha x)\, dx = \frac{3\left(\alpha^2 x^2 - 2\right)\sin(\alpha x)}{\alpha^4} - \frac{x\left(\alpha^2 x^2 - 6\right)\cos(\alpha x)}{\alpha^3}$$

$$\int x^4 \sin(\alpha x)\, dx = \frac{4x\left(\alpha^2 x^2 - 6\right)\sin(\alpha x)}{\alpha^4} - \frac{\left(\alpha^4 x^4 - 12\alpha^2 x^2 + 24\right)\cos(\alpha x)}{\alpha^5}$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arctan\left[\frac{x}{\sqrt{a^2 - x^2}}\right]\right)$$

$$\int_0^\infty x^{2k} e^{-\beta x^2} dx = \sqrt{\pi}\frac{(2k)!}{k!\, 2^{2k+1} \beta^{k+1/2}} \quad (\mathrm{Re}\, \beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^{2k+1} e^{-\beta x^2} dx = \frac{1}{2}\frac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\, \beta > 0,\ k = 0, 1, 2, ...)$$

$$\int_0^\infty x^k e^{-\gamma x} dx = \frac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\, \gamma > 0,\ k = 0, 1, 2, ...)$$

$$\int_{-\infty}^\infty e^{-\beta x^2} e^{iqx} dx = \sqrt{\frac{\pi}{\beta}} e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\, \beta > 0)$$

$$\int_0^\pi \sin^{2k} x\, dx = \pi\frac{(2k-1)!!}{2^k\, k!} \quad (k = 0, 1, 2, ...)$$

$$\int_0^\pi \sin^{2k+1} x\, dx = \frac{2^{k+1}\, k!}{(2k+1)!!} \quad (k = 0, 1, 2, ...)$$

$$\int_0^{2\pi} \cos m\phi\, e^{in\phi}\, dx = \pi(\delta_{m,n} + \delta_{m,-n}) \quad (m, n = 0, \pm 1, \pm 2, ...)$$

Useful Fourier integrals

$$\int \frac{1}{|\mathbf{r}|} e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r} = \frac{4\pi}{|\mathbf{q}|^2}$$

Useful trigonometric identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \qquad \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] \qquad \cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$

Useful identities for hyperbolic functions

$\cosh^2 x - \sinh^2 x = 1 \qquad \tanh^2 x + \mathrm{sech}^2\, x = 1 \qquad \coth^2 x - \mathrm{csch}^2\, x = 1$